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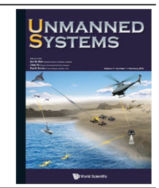


# Leader-Following Adaptive Consensus of Multiple Uncertain Rigid Body Systems over Jointly Connected Networks

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The leader-following consensus problem of a group of uncertain multiple rigid body systems subject to static networks has been solved by a distributed adaptive control law utilizing the distributed observer for the leader system. In this paper, we extend this result to jointly connected switching networks. This extension needs to overcome the discontinuity of some variables caused by the switching network. Additionally, we remove the assumption that every follower knows the system matrix of the leader system by employing an adaptive distributed observer for the leader system.

*Keywords*: Multi-agent systems; attitude consensus; adaptive control; parameter convergence.

## 1. Introduction

The consensus problem of multiple rigid body systems has received extensive attention. A typical application of this problem is the attitude synchronization of multiple spacecraft systems as studied in [10,13,15,18]. There are two types of attitude consensus problems, namely: leaderless consensus problem and leader-following consensus problem. The leader-following attitude consensus problem aims to design a distributed control law to drive all followers' attitudes and/or angular velocities to a class of prescribed trajectories generated by a so-called leader system. The problem was first studied under the assumption that all the followers know the leader's information directly in [1,9,20]. We call such a control law a purely decentralized control law. More recently, efforts have been made on designing a distributed control law to solve the same problem under various assumptions [3,5,11,14]. A distributed control law is a control law that satisfies the network communication constraint. In particular, by proposing a distributed observer for the leader system, [3] solved the leader-following attitude consensus problem provided that the communication network was static and connected. This technique in conjunction with the adaptive control technique has been further used to deal with the leader-following attitude consensus problem for multiple uncertain spacecraft systems in [6]. The control law in either [3] or [6] is not fully distributed in the sense that it assumes that all followers know the system matrix governing the angular velocity of the leader system. This assumption was removed in [4] by utilizing a so-called adaptive distributed observer for the leader system, which not only estimates the state of the leader system but also the system matrix of the leader system. The assumption that the communication network is static and connected may not be always satisfied in practice. A more practical scenario of the leader-following attitude consensus problem is that the communication network is switching and may be disconnected from time to time. References [11,12] studied this scenario by utilizing the distributed observer for the leader system and the adaptive distributed observer for the leader system, respectively.

In this paper, by combining the results and techniques in [6,12], we will consider the leader-following attitude consensus problem for multiple uncertain rigid body systems subject to jointly connected switching networks. Comparing with [6], the main challenge of this paper is to analyze the

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stability property of the closed-loop systems caused by the switching communication network. Specifically, a switching communication network will make the time derivative of some state variables discontinuous. Thus, the traditional Barbalat's lemma used in [6] is no longer applicable. Nevertheless, we manage to overcome this difficulty by invoking the generalized Barbalat's lemma in [16]. It is also worth noting that, unlike in [6], our control law is fully distributed since we employ the adaptive distributed observer as used in [12]. A preliminary version of this paper can be found in [17].

The remainder of this paper is organized as follows. In Sec. 2, we formulate our problem. Section 3 summarizes some existing results in [6,12]. In Sec. 4, we present our main results including the solvability of the problem and the convergence condition of the estimated unknown parameters to their actual values. In Sec. 5, we use two examples to illustrate our main results. Finally, we close this paper with some concluding remarks in Sec. 6.

In what follows, we use the unit quaternion to represent the attitude of the rigid body, which avoids the singularity. A quaternion $q = [\hat{q}^T \ \bar{q}]^T \in \mathbb{Q}$ denotes an element in $\mathbb{R}^4$, where $\hat{q} \in \mathbb{R}^3$ and $\bar{q} \in \mathbb{R}$. The conjugate of $q$ is given by $q^* = [-\hat{q}^T \ \bar{q}]^T$ and the inverse of $q$ is defined as $q^{-1} = \frac{q^*}{\|q\|}$ for $q \neq [0\ 0\ 0\ 0]^T$. If $\|q\| = 1$, then $q$ is said to be a unit quaternion. The set of unit quaternions is denoted by $\mathbb{Q}_u$. It is clear that for $q \in \mathbb{Q}_u$, $q^* = q^{-1}$. The addition and scalar multiplication of quaternions are the same as those of vectors. The multiplication of two quaternions $q_i, q_j \in \mathbb{Q}$ is defined as

$$q_i \odot q_j = \begin{bmatrix} \bar{q}_i\hat{q}_j + \bar{q}_j\hat{q}_i + \hat{q}_i^{\times}\hat{q}_j \\ \bar{q}_i\bar{q}_j - \hat{q}_i^T\hat{q}_j \end{bmatrix},$$

where, for $\hat{q}_i = [q_{i1}\ q_{i2}\ q_{i3}]^T$,

$$\hat{q}_i^{\times} = \begin{bmatrix} 0 & -q_{i3} & q_{i2} \\ q_{i3} & 0 & -q_{i1} \\ -q_{i2} & q_{i1} & 0 \end{bmatrix}.$$

For $x, y \in \mathbb{R}^3$, the following identities are useful:

$$\begin{aligned} &(x^{\times})^T = -x^{\times}, \quad x^T y^{\times} x = 0, \quad x^{\times} y = -y^{\times} x, \\ &x^{\times} x = 0, \quad x^T x^{\times} = 0, \quad (x+y)^{\times} = x^{\times} + y^{\times}, \\ &(x^{\times} y)^{\times} = yx^T - xy^T, \quad x^{\times} y^{\times} = yx^T - x^T y I_3. \end{aligned}$$

Let $q_I = [0\ 0\ 0\ 1]^T$. It can be verified that for $q \in \mathbb{Q}$, $q \odot q^{-1} = q^{-1} \odot q = q_I$ and $q \odot q_I = q_I \odot q = q$. In addition, the following properties can be easily checked and will be used in this paper: for $q_i, q_j, q_k \in \mathbb{Q}$, $q_i \odot (q_j + q_k) = q_i \odot q_j + q_i \odot q_k$, $(q_i \odot q_j) \odot q_k = q_i \odot (q_j \odot q_k)$, $(q_i \odot q_j)^* = q_j^* \odot q_i^*$. For $A_i \in \mathbb{R}^{n_i \times p}$, $i = 1, 2, \ldots, m$, $\mathrm{col}(A_1, \ldots, A_m) = [A_1^T \ \cdots \ A_m^T]^T$. $\boldsymbol{Q}(\alpha)$ where $\alpha \in \mathbb{R}^3$ denotes a vector in $\mathbb{R}^4$ in the form of $\mathrm{col}(\alpha, 0)$. $\mathrm{diag}(a_1, \ldots, a_n)$ denotes a diagonal matrix whose diagonal entries are $a_1, \ldots, a_n$, respectively. For $x = [x_1\ x_2\ x_3]^T \in \mathbb{R}^3$, a linear operator $L(\cdot)$ is defined as

$$L(x) = \begin{bmatrix} x_1 & 0 & 0 & 0 & x_3 & x_2 \\ 0 & x_2 & 0 & x_3 & 0 & x_1 \\ 0 & 0 & x_3 & x_2 & x_1 & 0 \end{bmatrix}.$$

## 2. Problem Formulation

As in [6], we consider a group of multiple rigid body systems whose $N$ followers take the following form:

$$\begin{aligned} \dot{q}_i &= \frac{1}{2} q_i \odot \boldsymbol{Q}(\omega_i), \\ J_i \dot{\omega}_i &= -\omega_i^{\times} J_i \omega_i + u_i, \quad i = 1, 2, \ldots, N, \end{aligned} \tag{1}$$

where $q_i \in \mathbb{Q}_u$ is the unit quaternion representation of the attitude of $i$th rigid body frame $\mathcal{B}_i$ with respect to the inertial frame $\mathcal{I}$. $\omega_i \in \mathbb{R}^3$ denotes the angular velocity of the $i$th rigid body frame with respect to $\mathcal{I}$. $J_i \in \mathbb{R}^{3\times 3}$ is the positive-definite inertia matrix of the $i$th rigid body and $u_i \in \mathbb{R}^3$ is its control input. It is noted that, for $i = 1, \ldots, N$, the entries of the matrices $J_i$ are unknown.

The leader system generating the reference attitude $q_0$ and the reference angular velocity $\omega_0 \in \mathbb{R}^3$ takes the following form:

$$\begin{aligned} \dot{v}_0 &= S v_0, \quad \omega_0 = E v_0, \\ \dot{q}_0 &= \frac{1}{2} q_0 \odot \boldsymbol{Q}(\omega_0), \end{aligned} \tag{2}$$

where $q_0 \in \mathbb{Q}_u$, $v_0 \in \mathbb{R}^n$, $\omega_0 \in \mathbb{R}^3$. $S \in \mathbb{R}^{n\times n}$ and $E \in \mathbb{R}^{3\times n}$ are constant matrices.

**Remark 2.1.** Without loss of generality, we assume that $q_i(0) \in \mathbb{Q}_u, i = 0, 1, 2, \ldots, N$. As was pointed out in [[5], Remark 2.2],

$$\frac{d}{dt}\|q_i(t)\|^2 = 0, \tag{3}$$

which means that if $q_i(0) \in \mathbb{Q}_u$, then $q_i(t) \in \mathbb{Q}_u$ for all $t \geq 0$.

Like in [3,5,11,12], the system composed of (1) and (2) can be viewed as a multi-agent system with $N+1$ agents. If all the followers can access the information of the leader, a decentralized control law which uses leader's information directly can be designed. However, for real application, this assumption is unrealistic because of the communication constraints. In this paper, we assume that the communication constraints are time-varying and described by digraph $\bar{\mathcal{G}}(t) = \{\bar{\mathcal{V}}, \bar{\mathcal{E}}(t)\}$, where $\bar{\mathcal{V}} = \{0, 1, 2, \ldots, N\}$ denotes the node set with node 0 associated with the leader, node $i$, $i = 1, 2, \ldots, N$, associated with the $i$th follower and $\bar{\mathcal{E}}(t) \subset$

$\bar{\mathcal{V}} \times \bar{\mathcal{V}}$ denotes the edge set. For $i = 0, 1, 2, \ldots, N$ and $j = 1, 2, \ldots, N$, $i \neq j$, $(i, j) \in \bar{\mathcal{E}}(t)$ if and only if the agent $j$ can access the information of the $i$th agent at time instant $t$ and we say $i$ is a neighbor of $j$ at time $t$. We use $\bar{\mathcal{N}}_i(t)$ to denote the set of all neighbors of node $i$ at time $t$. A subset of $\bar{\mathcal{E}}(t)$ in the form of $\{(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)\}$ is said to be a directed path of $\bar{\mathcal{G}}(t)$ from node $i_1$ to node $i_k$ at time $t$, and we say node $i_1$ can reach node $i_k$ at time $t$ if there is a directed path from $i_1$ to $i_k$ at time $t$. The digraph $\bar{\mathcal{G}}(t)$ is said to be static if $\bar{\mathcal{G}}(t) = \bar{\mathcal{G}}(0)$ for all $t > 0$. A time function $\sigma : [0, \infty) \mapsto \mathcal{P} = \{1, 2, \ldots, n_0\}$ with $n_0$ some positive integer is said to be a piecewise constant switching signal if there exists a sequence $\{t_j : j = 0, 1, 2, \ldots\}$ satisfying $t_0 = 0, t_{j+1} - t_j \geq \tau$ for some positive constant $\tau$ such that, for all $t \in [t_j, t_{j+1})$, $\sigma(t) = p$ for some $p \in \mathcal{P}$. $\mathcal{P}$ is called the switching index set, $t_j$ is called the switching instant, and $\tau$ is called the dwell time. Given a piecewise constant switching signal $\sigma(t)$ and a set of $n_0$ static digraphs $\bar{\mathcal{G}}_i = (\bar{\mathcal{V}}, \bar{\mathcal{E}}_i)$, $i = 1, \ldots, n_0$, we call the time-varying digraph $\bar{\mathcal{G}}_{\sigma(t)} = (\bar{\mathcal{V}}, \bar{\mathcal{E}}_{\sigma(t)})$ a switching digraph.

We need two standard assumptions as follows:

**Assumption 2.1.** There exists a subsequence $\{j_k : k = 0, 1, 2, \ldots\}$ of $\{j : j = 0, 1, 2, \ldots\}$ with $t_{j_{k+1}} - t_{j_k} < T$ for some positive $T$ such that node 0 can reach every other node in the union graph $\cup_{t \in [t_{j_k}, t_{j_{k+1}})} \bar{\mathcal{G}}_{\sigma(t)}$.

**Assumption 2.2.** All the eigenvalues of $S$ are semi-simple with zero real parts.

**Remark 2.2.** The above two assumptions are also needed in [11, 12]. Assumption 2.1 is called the jointly connected assumption [8]. It is the mildest assumption for the communication constraints since it allows the graph to be disconnected at any time instant. Assumption 2.2 guarantees the boundedness of $\omega_0(t)$. Under Assumption 2.2, the leader system (2) can generate a large class of reference angular velocities including step functions of any magnitude and sinusoidal functions of any magnitudes, any initial phases and any frequencies.

The control law we will design in this paper takes the following form:

$$\begin{aligned} u_i &= k_i(q_i, \omega_i, \phi_i, \psi_i), \\ \dot{\psi}_i &= f_i(q_i, \omega_i, \phi_i), \\ \dot{\phi}_i &= g_i(\phi_i, \phi_j - \phi_i, j \in \bar{\mathcal{N}}_i(t)), \quad i = 1, \ldots, N. \end{aligned} \tag{4}$$

where, for $i = 1, \ldots, N$, $u_i \in \mathbb{R}^3$. $k_i(\cdot)$, $f_i(\cdot)$ and $g_i(\cdot)$ are some smooth functions to be designed, $\phi_i$ is the state of the adaptive distributed observer to be introduced in the next section, $\psi_i$ provides an estimation of the entries of the matrices $J_i$, and $\phi_0$ consists of the components of $(q_0, \omega_0, S, E)$.

Also as in [6], we define the tracking error of attitude and angular velocity as follows:

$$\begin{aligned} \epsilon_i &= q_0^{-1} \odot q_i, \\ \tilde{\omega}_i &= \omega_i - C(\epsilon_i)\omega_0, \quad i = 1, 2, \ldots, N, \end{aligned} \tag{5}$$

where $C(\epsilon_i) = (\bar{\epsilon}_i^2 - \hat{\epsilon}_i^T \hat{\epsilon}_i) I_3 + 2\hat{\epsilon}_i \hat{\epsilon}_i^T - 2\bar{\epsilon}_i \hat{\epsilon}_i^\times$.

**Remark 2.3.** It can be easily verified that $\epsilon_i^T \epsilon_i = 1$. Therefore, for $i = 1, 2, \ldots, N$, $\epsilon_i \in \mathbb{Q}_u$. According to [[19], Proposition 1], the body frame $\mathcal{B}_i$ coincide with the body frame of the leader if and only if $\hat{\epsilon}_i = 0$.

Now we are ready to state our problem.

**Problem 2.1.** Given a multi-agent system composed of (1) and (2), a switching digraph $\bar{\mathcal{G}}_{\sigma(t)}$, design a distributed control law of the form (4) such that, for any initial conditions $q_i(0) \in \mathbb{Q}_u$, $\psi_i(0)$, $\phi_i(0)$, and $\omega_i(0) \in \mathbb{R}^3$, $i = 1, 2, \ldots, N$,

$$\lim_{t \to \infty} \hat{\epsilon}_i(t) = 0 \quad \text{and} \quad \lim_{t \to \infty} \tilde{\omega}_i(t) = 0.$$

**Remark 2.4.** The above problem can be viewed as a generalization of the problem in [6] from static networks to jointly connected networks, and is also a generalization of the problem in [12] from known rigid body systems to uncertain rigid body systems.

## 3. Preliminaries

Our control law will utilize the so-called adaptive distributed observer for the leader system first proposed in [4] and further enhanced in [12] as follows:

$$\dot{\eta}_i = \frac{1}{2} \eta_i \odot \boldsymbol{Q}(E_i v_i) + \mu_1 \sum_{j \in \bar{\mathcal{N}}_i(t)} (\eta_j - \eta_i), \tag{6}$$

$$\dot{v}_i = S_i v_i + \mu_2 \sum_{j \in \bar{\mathcal{N}}_i(t)} (v_j - v_i), \tag{7}$$

$$\dot{S}_i = \mu_3 \sum_{j \in \bar{\mathcal{N}}_i(t)} (S_j - S_i), \tag{8}$$

$$\dot{E}_i = \mu_4 \sum_{j \in \bar{\mathcal{N}}_i(t)} (E_j - E_i), \quad i = 1, 2, \ldots, N, \tag{9}$$

where, for $i = 1, 2, \ldots, N$, $\eta_i \in \mathbb{Q}$, $v_i \in \mathbb{R}^n$, $S_i \in \mathbb{R}^{n \times n}$ and $E_i \in \mathbb{R}^{3 \times n}$ are the $i$th agent's estimation of $q_0$, $v$, $S$ and $E$, respectively, and $\eta_0 = q_0$, $S_0 = S$ and $E_0 = E$.

It is shown in [12] that, under Assumptions 2.1 and 2.2, (6–9) have the property as stated in the following lemma.

**Lemma 3.1 ([[12], Sec. 3. A]).** *Under Assumptions 2.1 and 2.2, for any* $\mu_1, \mu_2, \mu_3, \mu_4 > 0$, *any initial conditions* $\eta_i(0)$, $v_i(0)$, $S_i(0)$ *and* $E_i(0)$, *and any* $q_0(t)$ *and* $v_0(t)$ *generated by* (2), *the solution of* (6) *to* (9) *satisfies*

$$\lim_{t\to\infty}(\eta_i(t) - q_0(t)) = 0,$$
$$\lim_{t\to\infty}(v_i(t) - v_0(t)) = 0,$$
$$\lim_{t\to\infty}(S_i(t) - S) = 0,$$
$$\lim_{t\to\infty}(E_i(t) - E) = 0,$$

*exponentially*, $i = 1, 2, \ldots, N$.

Like in [3,6], for $i = 1, 2, \ldots, N$, let

$$e_i = \eta_i^* \odot q_i, \tag{10}$$
$$\breve{\omega}_i = \omega_i - C(e_i)\zeta_i + k_{i1}\hat{e}_i, \tag{11}$$

where $\zeta_i = E_i v_i$, $e_i \in \mathbb{Q}$, $C(e_i) = (\bar{e}_i^2 - \hat{e}_i^T\hat{e}_i)I_3 + 2\hat{e}_i\hat{e}_i^T - 2\hat{e}_i\hat{e}_i^\times$, and $k_{i1} > 0$. Since, by Lemma 3.1, $\lim_{t\to\infty}(\eta_i^*(t) - q_0^{-1}(t)) = 0$, $\lim_{t\to\infty}(e_i(t) - \epsilon_i(t)) = 0$ and $\lim_{t\to\infty}(\tilde{\omega}_i(t) - \breve{\omega}_i(t)) = 0$. Thus, $e_i$ and $\breve{\omega}_i$ are the estimated tracking error of the attitude and the angular velocity, respectively.

It was shown in [3,6] that $e_i$ and $\breve{\omega}_i$ are governed by the following equations:

$$\dot{e}_i = \frac{1}{2}e_i \odot \boldsymbol{Q}(\breve{\omega}_i - k_{i1}\hat{e}_i) + e_{di}, \tag{12}$$
$$\begin{aligned} J_i\dot{\breve{\omega}}_i = {} & J_i((\breve{\omega}_i - k_{i1}\hat{e}_i)^\times C(e_i)E_i v_i - C(e_i)E_i S_i v_i \\ & + \frac{1}{2}k_{i1}(\bar{e}_i I_3 + \hat{e}_i^\times)(\breve{\omega}_i - k_{i1}\hat{e}_i)) \\ & - \omega_i^\times J_i\omega_i + u_i + J_{di}, \end{aligned} \tag{13}$$

where $e_{di} = \frac{1}{2}(e_i^T e_i - 1)\boldsymbol{Q}(\zeta_i) \odot e_i + \eta_{di}^* \odot q_i$ with $\eta_{di} = \mu_1\sum_{j\in\bar{\mathcal{N}}_i(t)}(\eta_j - \eta_i)$, and $J_{di} = -J_i C_{di}E_i v_i - J_i C(e_i)E_i v_{di} - J_i C(e_i)\dot{E}_i v_i + k_{i1}J_i\hat{e}_{di}$ with $C_{di} = 2\bar{e}_i\bar{e}_{di}I_3 - 2\hat{e}_i^T\hat{e}_{di}I_3 + 2\hat{e}_{di}\hat{e}_i^T + 2\hat{e}_i\hat{e}_{di}^T - 2\hat{e}_{di}\hat{e}_i^\times - 2\bar{e}_i\hat{e}_{di}^\times$ and $v_{di} = \mu_2\sum_{j\in\bar{\mathcal{N}}_i(t)}(v_j - v_i)$.

**Remark 3.1.** As pointed out in Remark 6 of [6], $\lim_{t\to\infty} e_i^T(t)e_i(t) \to 1$ exponentially. Thus, by Lemma 3.1, it can be easily verified that $e_{di}$, $\eta_{di}$, $J_{di}$, $C_{di}$ and $v_{di}$ all tend to zero exponentially as $t \to \infty$. Therefore, [[3], Lemma 4.1] can be invoked and is restated as follows.

**Lemma 3.2 ([[3], Lemma 4.1]).** *Consider system* (12). *If, for* $i = 1, 2, \ldots, N$, $\breve{\omega}_i(t)$ *is piecewise continuous over* $t \geq 0$ *and satisfies* $\lim_{t\to\infty}\breve{\omega}_i(t) = 0$, *then the solution of* (12) *is bounded for all* $t \geq 0$ *and* $\lim_{t\to\infty}\hat{e}_i(t) = 0$.

## 4. Main Results

In order to handle the unknown entries of the matrices $J_i$, $i = 1, \ldots, N$, by adaptive control technique, like in [2,6], for $x \in \mathbb{R}^3$, we make use of the linear operator $L(\cdot)$ introduced in Sec. 1. For $i = 1, 2, \ldots, N$, let

$$J_i = \begin{bmatrix} J_{i11} & J_{i12} & J_{i13} \\ J_{i12} & J_{i22} & J_{i23} \\ J_{i13} & J_{i23} & J_{i33} \end{bmatrix},$$

and

$$\Theta_i = \operatorname{col}(J_{i11}, J_{i22}, J_{i33}, J_{i23}, J_{i13}, J_{i12}).$$

It can be verified that, for any $x \in \mathbb{R}^3$, $J_i x = L(x)\Theta_i$. Therefore, (13) can be rewritten as follows:

$$J_i\dot{\breve{\omega}}_i = \chi_i\Theta_i + J_{di} + u_i, \tag{14}$$

where

$$\begin{aligned} \chi_i = {} & -\omega_i^\times L(\omega_i) + L((\breve{\omega}_i - k_{i1}\hat{e}_i)^\times C(e_i)E_i v_i \\ & - C(e_i)E_i S_i v_i + \frac{1}{2}k_{i1}(\hat{e}_i I_3 + \hat{e}_i^\times)(\breve{\omega}_i - k_{i1}\hat{e}_i)). \end{aligned}$$

Like [6], our control law takes the following form:

$$u_i = -\chi_i\hat{\Theta}_i - k_{i2}\breve{\omega}_i, \tag{15}$$

$$\dot{\hat{\Theta}}_i = \Lambda_i^{-1}\chi_i^T\breve{\omega}_i,\ i = 1, 2, \cdots, N, \tag{16}$$

where $\Lambda_i$ is some positive definite gain matrix and $k_{i2} > 0$.

**Remark 4.1.** The control law composed of (15) and (16) is somehow different from the control law in [6] since the matrices $E$ and $S$ in the control law of [6] are replaced with their estimations $E_i$ and $S_i$ in this paper. Thus, unlike in [6], our control law does not assume that every follower knows the matrices $S$ and $E$.

Now we are ready to state our first result.

**Theorem 4.1.** *Given systems* (1) *and* (2), *and a switched digraph* $\bar{\mathcal{G}}_{\sigma(t)}$, *under Assumptions 2.1 and 2.2, Problem 2.1 is solvable by the control law composed of* (6) *to* (9), (15) *and* (16).

**Proof.** The proof of this theorem is similar to that of [[6], Theorem 1]. Substituting (15) into (14) yields

$$J_i\dot{\breve{\omega}}_i = -\chi_i\tilde{\Theta}_i - k_{i2}\breve{\omega}_i + J_{di}, \tag{17}$$

where $\tilde{\Theta}_i = \hat{\Theta}_i - \Theta_i$. As in [6], we consider the following Lyapunov function candidate for (16) and (17)

$$V = \frac{1}{2}\sum_{i=1}^{N}(\breve{\omega}_i^T J_i\breve{\omega}_i + \tilde{\Theta}_i^T\Lambda_i\tilde{\Theta}_i). \tag{18}$$

By the same argument as used in the proof of Theorem 1 in [6], we conclude that, for $i = 1, 2, \ldots, N$, $\lim_{t\to\infty}\breve{\omega}_i(t) = 0$ and $\tilde{\Theta}_i$ is bounded. By Lemma 3.2, $\lim_{t\to\infty}\hat{e}_i(t) = 0$, and by Remark 3.1, $\lim_{t\to\infty}\hat{e}_i(t) = 1$. Thus, $\lim_{t\to\infty}e_i(t) = q_I$.

Since

$$\begin{aligned}\epsilon_i &= q_0^{-1} \odot q_i \\ &= \eta_i^* \odot q_i + (q_0 - \eta_i)^* \odot q_i \\ &= e_i + (q_0 - \eta_i)^* \odot q_i. \end{aligned} \tag{19}$$

By Lemma 3.1, and the fact that $\lim_{t\to\infty} e_i(t) = q_I$, we have $\lim_{t\to\infty}\epsilon_i(t) = q_I$. Hence, $\lim_{t\to\infty}\hat{\epsilon}_i(t) = 0$ and $\lim_{t\to\infty} C(e_i(t)) = \lim_{t\to\infty} C(\epsilon_i(t)) = I_3$.

Since

$$\begin{aligned}\tilde{\omega}_i &= \breve{\omega}_i + C(e_i)E_i v_i - C(\epsilon_i)Ev - k_{i1}\hat{e}_i \\ &= \breve{\omega}_i + C(e_i)E_i v_i - C(e_i)Ev + C(e_i)Ev \\ &\quad - C(\epsilon_i)Ev - k_{i1}\hat{e}_i \\ &= \breve{\omega}_i + C(e_i)(E_i v_i - Ev) \\ &\quad + (C(e_i) - C(\epsilon_i))Ev - k_{i1}\hat{e}_i, \end{aligned} \tag{20}$$

by Lemma 3.1 again, we have $\lim_{t\to\infty}\tilde{\omega}_i(t) = 0$. □

Even though $\hat{\Theta}_i$ is meant to estimate the unknown entries of $J_i$, Theorem 4.1 does not say anything about the convergence of the parameter $\hat{\Theta}_i$ to its actual value. In what follows, like [[6], Theorem 2], we will further establish a sufficient condition for the convergence of $\hat{\Theta}_i$ to its actual value. For this purpose, we will rephrase [[7], Lemma 2.4] as follows:

**Lemma 4.1.** *Let $f : [0,\infty) \to \mathbb{R}^n$ be bounded piecewise continuous function and $g : [0,\infty) \to \mathbb{R}^n$ be continuous differentiable function. Suppose $f(\cdot)$ satisfies the persistently exciting (PE) condition, i.e., there exist some $T_0, \alpha > 0$ such that*

$$\int_t^{t+T_0} f(\tau)f^T(\tau)d\tau \geq \alpha I, \tag{21}$$

*for all $t \geq 0$. Then*

$$\lim_{t\to\infty} g(t) = 0, \tag{22}$$

*if*

$$\lim_{t\to\infty} \dot{g}(t) = 0, \tag{23}$$

$$\lim_{t\to\infty} g^T(t)f(t) = 0. \tag{24}$$

**Remark 4.2.** By [[7], Lemma 2.2], the PE condition given in Lemma 4.1 is equivalent to the condition that there exist some $t_0, T_0, \mu > 0$ such that

$$\frac{1}{T_0}\int_t^{t+T_0} |c^T f(\tau)|d\tau \geq \mu, \tag{25}$$

for all $t \geq t_0$ and any unit vector $c \in \mathbb{R}^n$.

**Theorem 4.2.** *Let $\rho = \omega_0^T L(\dot{\omega}_0)$. If $\rho^T(t)$ satisfies the PE condition, then, for $i = 1, 2, \ldots, N$,*

$$\lim_{t\to\infty} \tilde{\Theta}_i(t) = 0. \tag{26}$$

**Proof.** First recall that a similar result was established in [[6], Theorem 2] for the static network case by using Barbalat's Lemma. In the case where the network is switching, the quantity $\dot{\breve{\omega}}_i$ of (17) is only piecewise continuous over $[0,\infty)$ since $J_{di}$ is discontinuous at the switching instants. Therefore, the Barbalat's Lemma as used in [6] cannot be used here anymore. We need to invoke the generalized Barbalat's Lemma in [16] to show that $\dot{\breve{\omega}}_i(t) \to 0$ as $t \to \infty$. For this purpose, we need to show the following:

(a) $\lim_{t\to\infty}\breve{\omega}_i(t)$ exists;
(b) $\breve{\omega}_i(t)$ is twice differentiable on each time interval $[t_j, t_{j+1})$, $j = 0, 1, 2, \ldots$;
(c) There exists a positive constant $K$ such that

$$\sup_{t\in[t_j,t_{j+1}), j=0,1,2,\ldots} |\ddot{\breve{\omega}}_i(t)| \leq K.$$

Part (a) has been established in Theorem 4.1. On each time interval $[t_j, t_{j+1})$, differentiating (17) with respect to time yields

$$\begin{aligned} J_i\ddot{\breve{\omega}}_i &= -\dot{\chi}_i\tilde{\Theta}_i - \chi_i\dot{\tilde{\Theta}}_i - k_{i2}\dot{\breve{\omega}}_{ii} + \dot{J}_{di} \\ &= -\dot{\chi}_i\tilde{\Theta}_i - \chi_i\Lambda_i^{-1}\chi_i^T\breve{\omega}_i \\ &\quad - k_{i2}J_i^{-1}(-\chi_i\tilde{\Theta}_i - k_{i2}\breve{\omega}_i + J_{di}) + \dot{J}_{di}. \end{aligned} \tag{27}$$

From the expression of $J_{di}$, it can be seen that $\dot{J}_{di}$ exists on each time interval $[t_j, t_{j+1})$. Thus, Part (b) is satisfied.

According to the second equation of (5), Assumption 2.2 and Theorem 4.1, $\omega_i$ is bounded. Since $\tilde{\Theta}_i$, $\omega_i$, $\breve{\omega}_i$, $e_i$, $C(e_i)$, $E_i$, $S_i$, $v_i$ are all bounded, from the expression of $\chi_i$, we have $\chi_i$ is bounded. And it can be verified that $\dot{\chi}_i$ is also bounded since $\omega_i$, $\dot{\omega}_i$, $\breve{\omega}_i$, $\dot{\breve{\omega}}_i$, $e_i$, $\dot{e}_i$, $E_i$, $\dot{E}_i$, $S_i$, $\dot{S}_i$, $v_i$, $\dot{v}_i$ are all bounded. In addition, $J_{di}$ is bounded from Remark 3.1 and $\dot{J}_{di}$ is also bounded since it is the combination of $\breve{\omega}_i$, $C(e_i)$, $\dot{C}(e_i)$, $\zeta_i$, $\dot{\zeta}_i$, $e_i$, $\dot{e}_i$, $\hat{\alpha}_i$, $\dot{\hat{\alpha}}_i$, $\eta_{di}$, $\dot{\eta}_{di}$, $q_i$, $E_i$, $\dot{E}_i$, $\ddot{E}_i$, $v_i$, $\dot{v}_i$, $v_{di}$ and $\dot{v}_{di}$, which are all bounded. Therefore, Part (c) is verified. By the generalized Barbalat's Lemma, $\lim_{t\to\infty}\dot{\breve{\omega}}_i(t) = 0$. From (17), we have

$$\lim_{t\to\infty} \chi_i(t)\tilde{\Theta}_i(t) = 0. \tag{28}$$

Then, as shown in details in [6], (28) implies the following:

$$\lim_{t\to\infty} \rho(t)\tilde{\Theta}_i(t) = \lim_{t\to\infty} \tilde{\Theta}_i^T(t)\rho^T(t) = 0. \tag{29}$$

From (16), noting that $\lim_{t\to\infty}\breve{\omega}_i(t) = 0$ and $\chi_i$ is bounded, we have

$$\lim_{t\to\infty} \dot{\tilde{\Theta}}_i(t) = 0. \tag{30}$$

Since $\rho^T(t)$ satisfies the PE condition, by (29), (30) and Lemma 4.1, we have, for $i = 1, 2, \ldots, N$,

$$\lim_{t\to\infty} \tilde{\Theta}_i(t) = 0, \tag{31}$$

which completes the proof. □

**Remark 4.3.** Theorem 4.2 states a sufficient condition of the convergence of parameter estimation error $\tilde{\Theta}_i(t)$. The importance of this condition will be revealed with an example in Sec. 5.

## 5. Two Examples

In this section, we will use two examples to illustrate the results in Section IV. Consider a group of multiple rigid body systems whose four followers take the form of (1) with $J_1 = \mathrm{diag}(1,2,3)$, $J_2 = \mathrm{diag}(1,4,5)$, $J_3 = \mathrm{diag}(1,6,7)$, $J_4 = \mathrm{diag}(1,8,9)$. The leader system takes the form of (2) with

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}, E = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This leader system will generate trigonometric polynomial with frequencies 0, 1 and 2.

The communication topology with $\mathcal{P} = \{1,2,3,4\}$ is illustrated by Fig. 1 where

$$\sigma(t) = \begin{cases} 1, & \text{if } kT_0 \le t < \left(k+\frac{1}{4}\right)T_0 \\ 2, & \text{if } \left(k+\frac{1}{4}\right)T_0 \le t < \left(k+\frac{1}{2}\right)T_0 \\ 3, & \text{if } \left(k+\frac{1}{2}\right)T_0 \le t < \left(k+\frac{3}{4}\right)T_0 \\ 4, & \text{if } \left(k+\frac{3}{4}\right)T_0 \le t < (k+1)T_0, \end{cases} \tag{32}$$

with $T_0 = 2$ and $k = 0,1,2,\ldots$.

It can be verified that both Assumptions 2.1 and 2.2 are satisfied. Thus, we can synthesize a control law composed of (6)–(9), (15) and (16).

To evaluate the performance of the control law, let various design parameters be $\mu_i = 10$. $k_{i1} = k_{i2} = 20$, $\Lambda_i = I_6$ for $i = 1,2,3,4$. The initial conditions are set as $q_0(0) = [0.5\ 0\ 0\ \frac{\sqrt{3}}{2}]^T$, $q_1(0) = [0\ 0\ 0\ 1]^T$, $q_2(0) = [1\ 0\ 0\ 0]^T$, $q_3(0) = [0\ 1\ 0\ 0]^T$, $q_4(0) = [0\ 0\ 1\ 0]^T$, $v_0(0) = [1\ 0\ 4\ 0\ 2]^T$, $\omega_1(0) = \omega_2(0) = \omega_3(0) = \omega_4(0) = [0\ 0\ 0]^T$. The other initial conditions are randomly chosen.

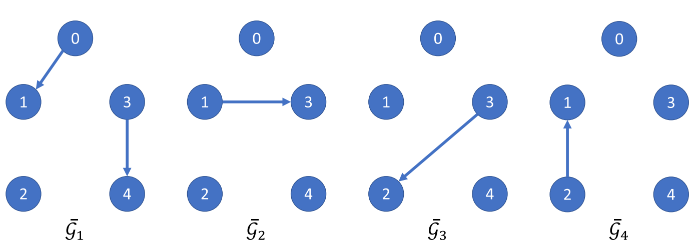


Fig. 1. Switching network topology $\bar{\mathcal{G}}_{\sigma(t)}$.

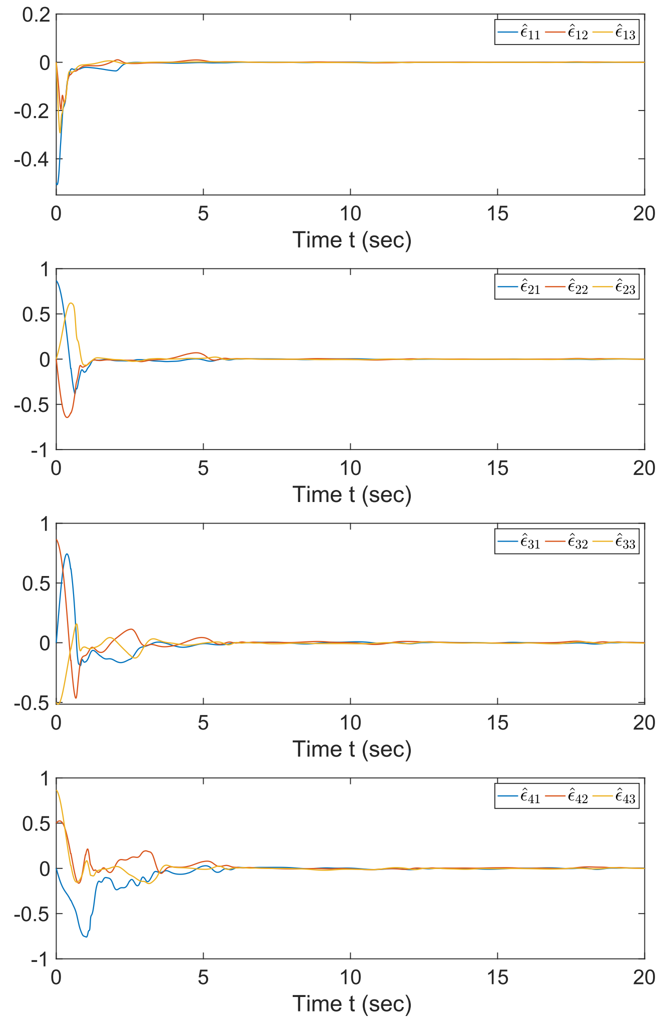


Fig. 2. Tracking errors of the attitude of the four followers with PE condition satisfied.

Figures 2 and 3 show the tracking errors of the attitude and angular velocity of the four followers, respectively. As expected, the tracking errors approach the origin asymptotically.

To verify whether the function $\rho^T(t)$ satisfies the PE condition, simple calculations give $\omega_0(t) = \mathrm{col}(1 + 4\sin t, 1 + 4\cos t, 2 + 2\cos(2t))$. Then $\rho = \omega_0^T L(\dot{\omega}_0) = \mathrm{col}(\rho_1, \ldots, \rho_6)^T$ with $\rho_1(t) = 4\cos t + 8\sin(2t)$, $\rho_2(t) = -4\sin t - 8\sin(2t)$, $\rho_3(t) = -8\sin(2t) - 4\sin(4t)$, $\rho_4(t) = 4\sin t - 8\sin(2t) + 4\sin(3t)$, $\rho_5(t) = 4\cos t + 12\cos(3t) - 8\sin(2t)$, $\rho_6(t) = 4\cos t + 16\cos(2t) - 4\sin t$. It can be seen that $\rho(t)$ is a periodic function with its period being $2\pi$. Thus, $\rho(t)^T\rho(t)$ is a periodic signal with period $2\pi$.

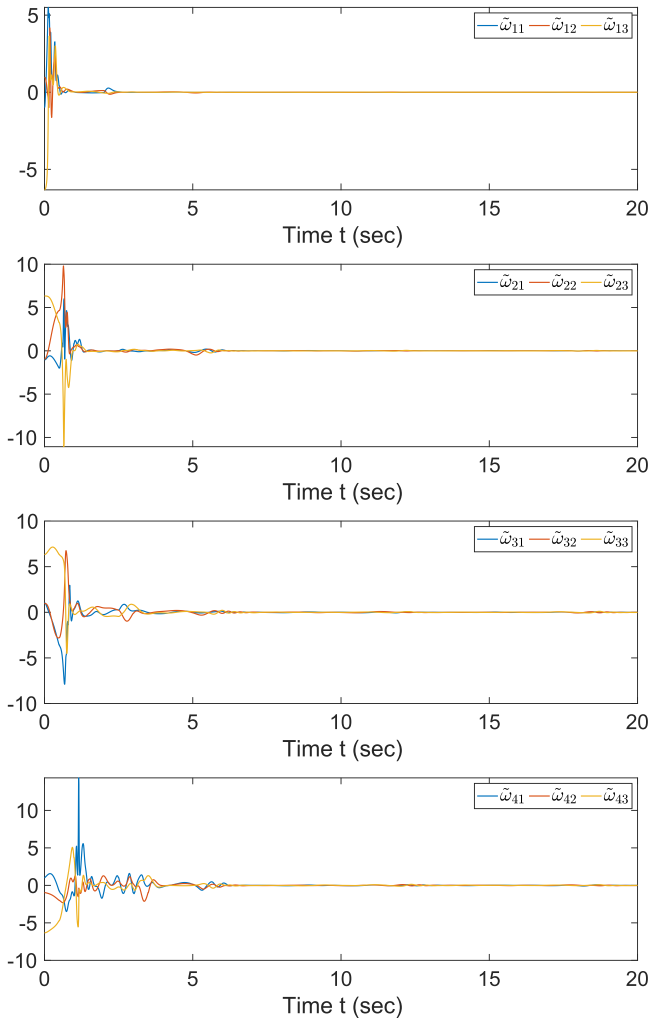


Fig. 3. Tracking errors of the angular velocity of the four followers with PE condition satisfied.

Let $T_0 = 2\pi$ in (21), by periodicity, the integration from $t$ to $t + 2\pi$ is identical for any $t$. Simple calculation gives

$$\int_0^{2\pi} \rho^T(\tau)\rho(\tau)d\tau = \begin{bmatrix} 80\pi & -32\pi & -64\pi & -64\pi & -48\pi & 16\pi \\ -32\pi & 80\pi & 64\pi & 48\pi & 64\pi & 16\pi \\ -64\pi & 64\pi & 80\pi & 64\pi & 64\pi & 0 \\ -64\pi & 48\pi & 64\pi & 96\pi & 64\pi & -16\pi \\ -48\pi & 64\pi & 64\pi & 64\pi & 224\pi & 16\pi \\ 16\pi & 16\pi & 0 & -16\pi & 16\pi & 288\pi \end{bmatrix} > 10I. \tag{33}$$

Hence, $\rho^T(t)$ satisfies the PE condition. Figure 4 shows the convergence of all the estimated unknown parameters to their actual values.

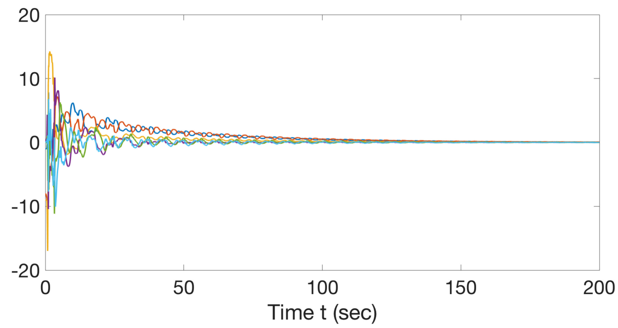


Fig. 4. Estimation errors of uncertain parameters with PE condition satisfied.

To see the importance of the PE condition, let us change the $E$ matrix to the following:

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

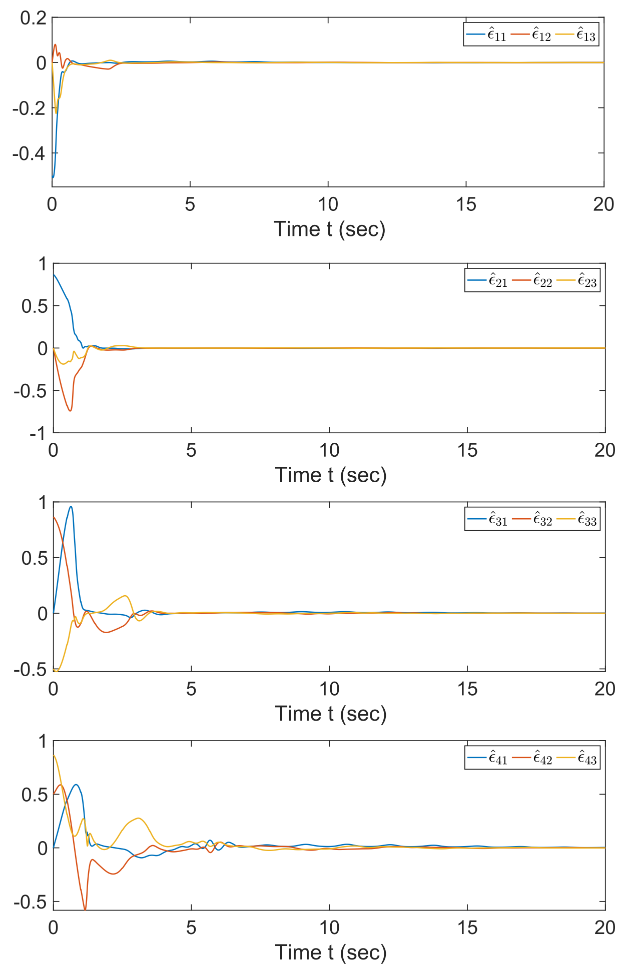


Fig. 5. Tracking errors of the attitude of the four followers with PE condition unsatisfied.

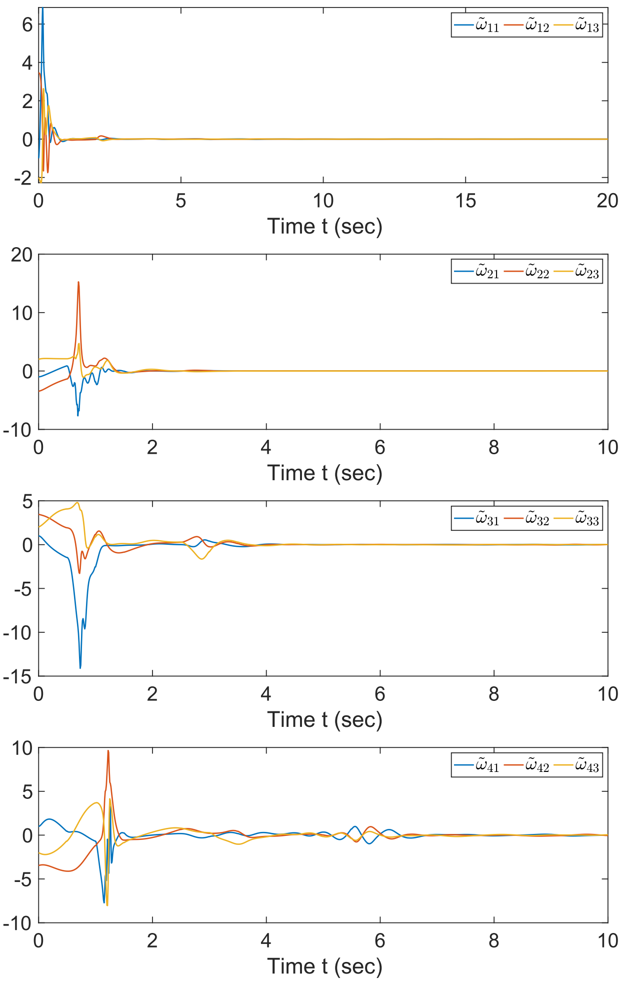


Fig. 6. Tracking errors of the angular velocity of the four followers with PE condition unsatisfied.

All other parameters and initial conditions are the same as before. In this case, we have $\omega_0(t) = \mathrm{col}(1, 4\sin t, 4\cos t)$ and $\rho(t) = \mathrm{col}(\rho_1, \ldots, \rho_6)^T$ with $\rho_1 = 0$, $\rho_2 = 8\sin(2t)$, $\rho_3 = -8\sin(2t)$, $\rho_4 = 16\cos^2 t - 16\sin^2 t$, $\rho_5 = -4\sin t$,

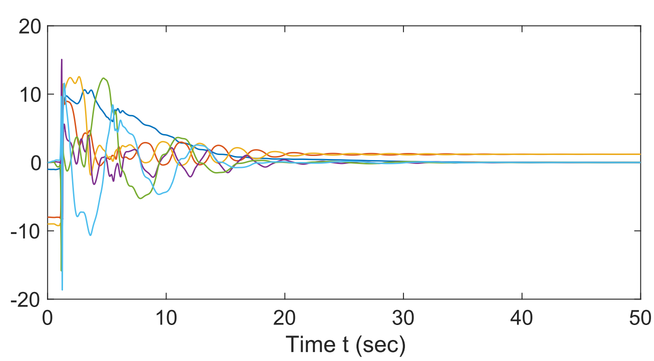


Fig. 7. Estimation errors of uncertain parameters with PE condition unsatisfied.

$\rho_6 = 4\cos t$. Let $c = [\frac{\sqrt{3}}{3}\ \frac{\sqrt{3}}{3}\ -\frac{\sqrt{3}}{3}\ 0\ 0\ 0]^T$. Then it can be verified that $c^T\rho^T(t) = 0$ for all $t$. Thus, by Remark 4.2, $\rho^T(t)$ does not satisfy the PE condition. Figures 5 and 6 show the tracking errors of the attitude and angular velocity of the four followers, respectively. It can be seen that the control law can still drive all the tracking errors to the origin asymptotically. Nevertheless, as shown in Fig. 7, not all the estimated unknown parameters converge to their actual values.

## 6. Conclusion

In this paper, we have studied the leader-following attitude consensus problem of a group of multiple uncertain rigid body systems subject to jointly connected switching communication networks, which can be viewed as a generalization of the result in [6]. To achieve this extension, we have overcome the difficulty of the discontinuity of the derivatives of $\tilde\omega$ in Theorem 4.2 by making use of the generalized Barbalat's Lemma. Moreover, by employing an adaptive distributed observer for the leader system, we have removed the assumption that every follower knows the system matrices of the leader system in [6].

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