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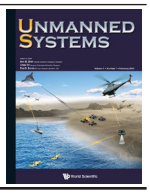

Consensus of Multi-Agent Systems with Control-Affine Nonlinear Dynamics

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This paper studies the consensus problem for a class of multi-agent systems, where each agent is a nonlinear system affine in control. Distributed control laws are proposed for multi-agent systems with and without communication delays, respectively. It is shown that under undirected and connected communication network our control laws ensure the state consensus of multi-agent systems in both cases. The effectiveness of the proposed control laws is demonstrated by numerical simulation.

Keywords: Consensus; nonlinear systems; time delays.

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1. Introduction

The collective motion of agents in a large community with a common objective, such as flocking of birds and schooling of fishes, is easy to observe in nature. This cooperative behavior has been studied extensively by scientists and engineers for its potential in applications such as coordination of robots, cooperation of unmanned air vehicles, and traffic control.

Much of the existing research on cooperative control addresses the consensus problem. Consensus means that all the interacting agents reach an agreement on their states. One can refer to [1, 2] for overviews of research on the consensus problem.

Many researchers have studied the consensus problem of multi-agent systems with linear models. [3] studied a double integer agent model and delayed directed networks. Consensus of higher order multi-agent systems was studied in [4, 5]. In [4], a truncated predictor feedback-based protocol was developed for multi-agent systems with bounded communication delays. It was shown that the closed-loop system tolerates arbitrarily large communication delays. The work conducted in [5] also allowed the presence of internal uncertainties and external disturbances.

More recently, there has been a surge of interest in consensus of nonlinear multi-agent systems. In [6], the second-order consensus problem was studied for multi-agent systems whose agent dynamics are in a feedback linearizable form. Reference [7] addressed leader-following consensus of multi-agent systems whose dynamics are given in a normal form with uncertainties. Two neural networks were employed to estimate nonlinearities in the system. In [8], the author investigated output consensus of a class of affine nonlinear systems that possess input-output passivity. It was demonstrated that output consensus is guaranteed if the storage function of the system is positive definite and radially unbounded. Flocking of multiple vehicles with nonholonomic dynamics was studied in [9], where communication delays and collision avoidance were taken into consideration as well, and a potential functional-based low gain control law was proposed. It was shown that the closed-loop multi-agent system tolerates arbitrarily large communication delays.

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In this paper, we study the consensus problem of multi-agent systems with nonlinear affine-in-control dynamics. Communication delays are considered and the low gain design method is used to eliminate their effects. The upper bounds of the tolerable delays and the corresponding low gain parameter are determined explicitly. We demonstrate that the proposed control laws bring the multi-agent system into consensus if communication delay is within the derived bound and the low gain parameter is small enough. The stabilization problem for such control-affine nonlinear systems has been studied in [10].

The remainder of this paper is organized as follows. After introducing necessary preliminaries in Sec. 2, we give the problem definition and assumptions in Sec. 3. Section 4 presents the distributed control laws and the analysis of the multi-agent system under the proposed distributed control laws. Section 5 considers the situation when the control inputs are subject to communication delays. In Sec. 6, the effectiveness of the proposed control laws is demonstrated by numerical simulation. Section 7 draws the conclusion to the paper.

2. Preliminaries

A. *Graph theory*

Definition 1. An undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ consists of a nonempty set of nodes $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$, and a set of edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. An unordered pair $(v_i, v_j) \in \mathcal{E}$ if there exists a bidirectional communication link between node v_i and node v_j.

Definition 2. The topology of an undirected graph $\mathcal{G}$ can be described by an associated adjacency matrix $A_{\text{adj}} = [a_{ij}] \in \mathbb{R}^{N\times N}$, which is defined as

$$a_{ij} = a_{ji} = \begin{cases} 1, & \text{if } (v_i, v_j) \in \mathcal{E}, \\ 0, & \text{otherwise.} \end{cases}$$

We define the graph Laplacian matrix $L = D - A_{\text{adj}}$, where $D = \text{diag}\{d_1, d_2, \ldots, d_N\}$ and $d_i = \sum_{j=1}^{N} a_{ij}, i = 1, 2, \ldots, N$. For an undirected graph, $L = L^{\mathrm{T}}$.

Lemma 1 ([1]). *The Laplacian matrix L of an undirected graph is positive semi-definite.*

Lemma 2 ([11]). *An undirected graph is connected if and only if zero is a simple eigenvalue of the Laplacian Matrix L with* $1_{N\times 1} = [1, 1, \ldots, 1]^{\mathrm{T}} \in \mathbb{R}^N$ *being the only corresponding eigenvector.*

B. *Lie algebra*

Let $f(z) : \mathbb{R}^n \to \mathbb{R}^n$ and $h(z) : \mathbb{R}^n \to \mathbb{R}^n$ be smooth vector fields. Their Lie bracket is defined by

$$[f(z), h(z)] = \frac{\partial h}{\partial z} f(z) - \frac{\partial f}{\partial z} h(z),$$

and we define following notations inductively

$$\begin{aligned} &\text{ad}_f^0 h(z) = h(z), \\ &\text{ad}_f^q h(z) = [f(z),\ \text{ad}_f^{q-1} h(z)],\ q \in \mathbb{N}^+. \end{aligned}$$

The Lie derivatives are also defined inductively by

$$\begin{aligned} &L_f^0 h(z) = h(z), \\ &L_f^q h(z) = \frac{\partial L_f^{q-1} h(z)}{\partial z} f(z), \quad q \in \mathbb{N}^+. \end{aligned}$$

3. Problem Statement

We consider a multi-agent system consisting of N agents, each described by,

$$\dot{x}_i = f(x_i) + g(x_i) u_i, \quad i = 1, 2, \ldots, N, \tag{1}$$

where $x_i \in \mathbb{R}^n$ is the state vector of agent i, $u_i \in \mathbb{R}^m$ is the control input, $g(x_i) = [g_1(x_i), g_2(x_i), \ldots, g_m(x_i)] \in \mathbb{R}^{n\times m}$, and $f(x_i)$ and $g_k(x_i)$, $k = 1, 2, \ldots, m$, are smooth vector fields in $\mathbb{R}^n$.

Definition 3. Global consensus of a multi-agent system is said to be achieved if the states of all agents are synchronized, i.e., for all $x_i(0) \in \mathbb{R}^n$, $i = 1, 2, \ldots, N$, $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0, i, j = 1, 2, \ldots, N$.

Neighborhood consensus errors are defined as

$$e_i = \sum_{j=1}^{N} a_{ij}(x_i - x_j), \quad i = 1, 2, \ldots, N. \tag{2}$$

We introduce the following notations,

$$x = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, \ldots, x_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nn}, \tag{3}$$

$$e = [e_1^{\mathrm{T}}, e_2^{\mathrm{T}}, \ldots, e_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nn}, \tag{4}$$

$$u = [u_1^{\mathrm{T}}, u_2^{\mathrm{T}}, \ldots, u_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nm}, \tag{5}$$

$$F(x) = [f^{\mathrm{T}}(x_1), f^{\mathrm{T}}(x_2), \ldots, f^{\mathrm{T}}(x_N)]^{\mathrm{T}} \in \mathbb{R}^{Nn}, \tag{6}$$

$$G(x) = \begin{bmatrix} g(x_1) & & & \\ & g(x_2) & & \\ & & \ddots & \\ & & & g(x_N) \end{bmatrix} \in \mathbb{R}^{Nn\times Nm}. \quad (7)$$

Then the dynamics of the multi-agent system (1) can be expressed more compactly as

$$\dot{x} = F(x) + G(x)u. \quad (8)$$

The consensus errors (2) can be rewritten as

$$e = (L \otimes I_n)x, \quad (9)$$

$$\dot{e} = (L \otimes I_n)(F(x) + G(x)u). \quad (10)$$

Assumption 1. The communication topology is described by an undirected and connected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V}$ is the indexed set of agents in the system and $\mathcal{E}$ represents the communication links among the agents.

Remark 1. The assumption of an undirected and connected graph indicates that consensus is achieved if and only if all the elements in $e(t)$ approach zero as t goes to infinity. Since the Laplacian matrix L is symmetric, there exists some orthogonal matrix $T \in \mathbb{R}^{N\times N}$ such that

$$T^{\mathrm{T}}LT = \begin{bmatrix} 0 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{bmatrix} \triangleq J_L.$$

By Lemma 1, we know that L is positive semi-definite and $\lambda_2, \lambda_3, \ldots, \lambda_N > 0$. Therefore, we can always construct a positive definite matrix M by $M = TJ_MT^{\mathrm{T}}$, where

$$J_M = \begin{bmatrix} \frac{1}{\lambda_1} & & & \\ & \frac{1}{\lambda_2} & & \\ & & \ddots & \\ & & & \frac{1}{\lambda_N} \end{bmatrix},$$

with any $\lambda_1 > 0$. Noting that $J_LJ_MJ_L = J_L$, we have

$$\begin{aligned} LML &= TJ_LT^{\mathrm{T}}TJ_MT^{\mathrm{T}}TJ_LT^{\mathrm{T}} \\ &= TJ_LT^{\mathrm{T}} \\ &= L. \end{aligned}$$

Consequently, we have

$$\begin{aligned} e^{\mathrm{T}}(ML \otimes I_n) &= x^{\mathrm{T}}(LML \otimes I_n) \\ &= x^{\mathrm{T}}(L \otimes I_n) \\ &= e^{\mathrm{T}}. \end{aligned} \quad (11)$$

Assumption 2. There exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$, such that for all $x_i, x_j \in \mathbb{R}^n$,

$$(x_i - x_j)^{\mathrm{T}}P[f(x_i) - f(x_j)] \le 0.$$

Remark 2. Suppose that system (1) has linear unforced dynamics, that is, $f(x_i) = Ax_i$, $i = 1, 2, \ldots, N$. If each individual system is open-loop marginally stable, we can find a matrix $P > 0$ which satisfies Assumption 2 by solving the Lyapunov equation $A^{\mathrm{T}}P + PA = Q$, for some $Q \le 0$. For systems that have nonlinear $f(x_i)$, this assumption requires the entire multi-agent system to be "open-loop cooperatively marginally stable." That is, in the absence of control inputs the distance between any two agents remains bounded.

Assumption 3. For any $z \in \mathbb{R}^n$, we have

$$\dim \operatorname{span}\{\operatorname{ad}_f^q g_k(z) : 0 \le q \le n-1,\ 1 \le k \le m\} = n.$$

Remark 3. Assumption 3 is a controllability-like condition. If an affine in control system is open-loop marginally stable and this controllability-like condition is satisfied, then the system is globally asymptotically stabilizable [10].

4. Global Consensus Algorithms

In this section, we propose the following distributed control laws,

$$\begin{aligned} u_i(t) &= -g^{\mathrm{T}}(x_i(t))Pe_i(t) \\ &= -g^{\mathrm{T}}(x_i(t))P\sum_{j=1}^{N} a_{ij}(x_i(t) - x_j(t)), \\ &\qquad i = 1, 2, \ldots, N, \end{aligned} \quad (12)$$

where the definition of the matrix P is as defined in Assumption 2. Using the notations in (3)–(7), we obtain

$$u(t) = -G^{\mathrm{T}}(x(t))(I_N \otimes P)e(t). \quad (13)$$

The following three lemmas are important in the analysis of the closed-loop multi-agent system. Lemma 3 states a useful property of the Laplacian matrix. Lemma 4 implies that, under Assumption 2, the consensus errors remain bounded in the absence of control inputs. Lemma 5 is a consequence of the controllability-like condition, as introduced in Assumption 3, to the overall multi-agent system.

Lemma 3. *Let $L = L^{\mathrm{T}}$ be the Laplacian matrix of an undirected graph $\mathcal{G}$. For any $\upsilon = [\upsilon_1^{\mathrm{T}}, \upsilon_2^{\mathrm{T}}, \ldots, \upsilon_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nn}$*

and $\omega = [\omega_1^{\mathrm{T}}, \omega_2^{\mathrm{T}}, \ldots, \omega_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nn}$*, we have*

$$2\upsilon^{\mathrm{T}}(L \otimes I_n)\omega = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(\upsilon_i - \upsilon_j)^{\mathrm{T}}(\omega_i - \omega_j),$$

where a_{ij} *is the entry of the adjacency matrix of the undirected graph* $\mathcal{G}$*.*

Proof. By the definition of the Laplacian matrix and straightforward calculations, we obtain

$$\begin{aligned}\upsilon^{\mathrm{T}}(L \otimes I_n)\omega &= \sum_{i=1}^{N} \upsilon_i^{\mathrm{T}} \sum_{j=1}^{N} a_{ij}(\omega_i - \omega_j) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\upsilon_i^{\mathrm{T}}(\omega_i - \omega_j). \qquad (14)\end{aligned}$$

For an undirected graph, $a_{ij} = a_{ji}$. Thus,

$$\upsilon^{\mathrm{T}}(L \otimes I_n)\omega = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ji}\upsilon_i^{\mathrm{T}}(\omega_i - \omega_j).$$

Reordering and renaming the summation indices, we have

$$\upsilon^{\mathrm{T}}(L \otimes I_n)\omega = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\upsilon_j^{\mathrm{T}}(\omega_j - \omega_i). \qquad (15)$$

Adding both sides of (14) and (15) gives

$$\begin{aligned}2\upsilon^{\mathrm{T}}(L \otimes I_n)\omega &= \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\upsilon_i^{\mathrm{T}}(\omega_i - \omega_j) \\ &\quad + \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\upsilon_j^{\mathrm{T}}(\omega_j - \omega_i) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(\upsilon_i - \upsilon_j)^{\mathrm{T}}(\omega_i - \omega_j).\end{aligned}$$

□

Lemma 4. *If Assumption* 2 *is satisfied, then,*

$$e^{\mathrm{T}}(I_N \otimes P)F(x) \le 0,$$

where the matrix $P = P^{\mathrm{T}}$ *is as defined in Assumption* 2*.*

Proof. We note that

$$\begin{aligned}(I_N \otimes P)F(x) &= [f^{\mathrm{T}}(x_1)P, f^{\mathrm{T}}(x_2)P, \ldots, f^{\mathrm{T}}(x_N)P]^{\mathrm{T}} \\ &\triangleq F_P(x).\end{aligned}$$

Then

$$\begin{aligned}e^{\mathrm{T}}(I_N \otimes P)F(x) &= e^{\mathrm{T}}F_P(x) \\ &= x^{\mathrm{T}}(L \otimes I_n)F_P(x). \qquad (16)\end{aligned}$$

Applying Lemma 3 to (16) gives

$$e^{\mathrm{T}}(I_N \otimes P)F(x) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(x_i - x_j)^{\mathrm{T}}P(f(x_i) - f(x_j)).$$

By Assumption 2, we have $e^{\mathrm{T}}(I_N \otimes P)F(x) \le 0$. □

Lemma 5. *If Assumption* 3 *is satisfied, then, for any* $x \in \mathbb{R}^{Nn}$*,* $\dim \operatorname{span}\{\mathrm{ad}_F^q G_r(x) : 0 \le q \le n-1, 1 \le r \le Nm\} = Nn$*, where* $G_r(x)$ *denotes the rth column of* $G(x)$*.*

Proof. We define the following notations,

$$\mathbf{0}_{n\times 1} = \begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix} \in \mathbb{R}^n, \quad \mathbf{0}_{n\times n} = \begin{bmatrix}0 & 0 & \cdots & 0\\0 & 0 & \cdots & 0\\ & & & \\ \vdots & \vdots & \ddots & \vdots\\0 & 0 & \cdots & 0\end{bmatrix} \in \mathbb{R}^{n\times n}.$$

Then for all $i = 1, 2, \ldots, N$, $k = 1, 2, \ldots, m$,

$$G_{(i-1)m+k}(x) = \begin{bmatrix}\mathbf{0}_{n\times 1}\\\vdots\\\mathbf{0}_{n\times 1}\\g_k(x_i)\\\mathbf{0}_{n\times 1}\\\vdots\\\mathbf{0}_{n\times 1}\end{bmatrix} \left.\begin{matrix}\\\\\\\\\\\\\end{matrix}\right\} i-1.$$

By the definition of Lie brackets, we have

$$\mathrm{ad}_F^0 G_{(i-1)m+k}(x) = \begin{bmatrix}\mathbf{0}_{n\times 1}\\\vdots\\\mathbf{0}_{n\times 1}\\g_k(x_i)\\\mathbf{0}_{n\times 1}\\\vdots\\\mathbf{0}_{n\times 1}\end{bmatrix} \left.\begin{matrix}\\\\\\\\\\\\\end{matrix}\right\} i-1,$$

$$\mathrm{ad}_F G_{(i-1)m+k}(x)$$

$$= \begin{bmatrix}\mathbf{0}_{n\times n} & \cdots & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \cdots & \mathbf{0}_{n\times n}\\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \mathbf{0}_{n\times n} & \cdots & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \cdots & \mathbf{0}_{n\times n}\\ \mathbf{0}_{n\times n} & \cdots & \mathbf{0}_{n\times n} & \frac{\partial_{gk}(x_i)}{\partial_{x_i}} & \mathbf{0}_{n\times n} & \cdots & \mathbf{0}_{n\times n}\\ \mathbf{0}_{n\times n} & \cdots & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \cdots & \mathbf{0}_{n\times n}\\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \underbrace{\mathbf{0}_{n\times n} \cdots \mathbf{0}_{n\times n}}_{i-1} & & & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \cdots & \mathbf{0}_{n\times n}\end{bmatrix} \begin{bmatrix}f(x_1)\\\vdots\\f(x_i)\\\vdots\\f(x_N)\end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial f(x_1)}{\partial x_1} & & & \\ & \frac{\partial f(x_2)}{\partial x_2} & & \\ & & \ddots & \\ & & & \frac{\partial f(x_N)}{\partial x_N} \end{bmatrix} \begin{bmatrix} \mathbf{0}_{n\times 1} \\ \vdots \\ \mathbf{0}_{n\times 1} \\ g_k(x_i) \\ \mathbf{0}_{n\times 1} \\ \vdots \\ \mathbf{0}_{n\times 1} \end{bmatrix} \Bigg\} \; i-1$$

$$= \begin{bmatrix} \mathbf{0}_{n\times 1} \\ \vdots \\ \mathbf{0}_{n\times 1} \\ \frac{\partial_{gk}(x_i)}{\partial_{x_i}} f(x_i) \\ \mathbf{0}_{n\times 1} \\ \vdots \\ \mathbf{0}_{n\times 1} \end{bmatrix} - \begin{bmatrix} \mathbf{0}_{n\times 1} \\ \vdots \\ \mathbf{0}_{n\times 1} \\ \frac{\partial_{gk}(x_i)}{\partial_{x_i}} g_k(x_i) \\ \mathbf{0}_{n\times 1} \\ \vdots \\ \mathbf{0}_{n\times 1} \end{bmatrix} \Bigg\} \; i-1$$

$$= -\begin{bmatrix} \mathbf{0}_{n\times 1} \\ \vdots \\ \mathbf{0}_{n\times 1} \\ ad_f g_k(x_i) \\ \mathbf{0}_{n\times 1} \\ \vdots \\ \mathbf{0}_{n\times 1} \end{bmatrix} \Bigg\} \; i-1.$$

Similarly, it is straightforward to show that, for all $i = 1, 2, \ldots, N$, $k = 1, 2, \ldots, m$, $q \in \mathbb{N}$,

$$ad_F^q G_{(i-1)m+k}(x) = \begin{bmatrix} \mathbf{0}_{n\times 1} \\ \vdots \\ \mathbf{0}_{n\times 1} \\ ad_F^q g_k(x_i) \\ \mathbf{0}_{n\times 1} \\ \vdots \\ \mathbf{0}_{n\times 1} \end{bmatrix} \Bigg\} \; i-1. \tag{17}$$

By Assumption 3 and (17), we have, for any $x \in \mathbb{R}^{Nn}$,

$$\dim \operatorname{span}\{\operatorname{ad}_F^q G_r(x) : 0 \le q \le n-1, 1 \le r \le Nm\} = Nn.$$

□

The behavior of the multi-agent system (1) under the proposed control laws (12) is established in the following theorem.

Theorem 1. *Consider a multi-agent system whose dynamics are given by* (1). *Suppose that Assumptions* 1–3 *hold. Then the control laws* (12) *steer the multi-agent system into global consensus. That is,*

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0, \quad i, j = 1, 2, \ldots, N.$$

Proof. Consider the following Lyapunov function

$$V_{\mathrm{M}}(e) = e^{\mathrm{T}}(t)(M \otimes P)e(t),$$

where matrices $M > 0$ and $P > 0$ are defined in Remark 1 and Assumption 2, respectively. For brevity, we drop all the time variables in the remainder of the proof.

Taking time derivative along the trajectories of (10) yields

$$\begin{aligned} \dot{V}_{\mathrm{M}} &= 2e^{\mathrm{T}}(M \otimes P)\dot{e} \\ &= 2e^{\mathrm{T}}(M \otimes P)(L \otimes I_n)(F(x) + G(x)u) \\ &= 2e^{\mathrm{T}}(ML \otimes I_n)(I_N \otimes P)(F(x) + G(x)u). \end{aligned} \tag{18}$$

In view of (11), we have

$$\dot{V}_{\mathrm{M}} = 2e^{\mathrm{T}}(I_N \otimes P)(F(x) + G(x)u). \tag{19}$$

Substituting (13) into (19) gives

$$\begin{aligned} \dot{V}_{\mathrm{M}} = {} & 2e^{\mathrm{T}}(I_N \otimes P)F(x) \\ & -2e^{\mathrm{T}}(I_N \otimes P)G(x)G^{\mathrm{T}}(x)(I_N \otimes P)e. \end{aligned} \tag{20}$$

By Lemma 4, we conclude that $\dot{V}_{\mathrm{M}} \le 0$.

Since both terms in the right-hand side of (20) are nonpositive, $\dot{V}_{\mathrm{M}} = 0$ if and only if $2e^{\mathrm{T}}(I_N \otimes P)F(x) = 0$ and $2e^{\mathrm{T}}(I_N \otimes P)G_r(x) = 0, 1 \le r \le Nm$, which are respectively equivalent to

$$2x^{\mathrm{T}}(L \otimes P)F(x) = 0 \tag{21}$$

and

$$2x^{\mathrm{T}}(L \otimes P)G_r(x) = 0, \quad 1 \le r \le Nm. \tag{22}$$

Note that $V_{\mathrm{M}}(e) = x^{\mathrm{T}}(LML \otimes P)x = x^{\mathrm{T}}(L \otimes P)x$. Then,

$$\frac{\partial V_{\mathrm{M}}(e)}{\partial x} = 2x^{\mathrm{T}}(L \otimes P). \tag{23}$$

Substitutions of (23) into (21) and (22) result in

$$\frac{\partial V_{\mathrm{M}}(e)}{\partial x} F(x) = 0 \tag{24}$$

and

$$\frac{\partial V_{\mathrm{M}}(e)}{\partial x} G_r(x) = 0, \quad 1 \le r \le Nm, \tag{25}$$

respectively. It is clear now $\dot{V}_{\mathrm{M}} = 0$ if and only if

$$L_F V_{\mathrm{M}}(e) = 0 \tag{26}$$

and

$$L_{G_r} V_{\mathrm{M}}(e) = 0, \quad 1 \le r \le Nm. \tag{27}$$

Here we have abused the notations by using the Lie derivative notations $L_F V_{\mathrm{M}}(e)$ and $L_{G_r} V_{\mathrm{M}}(e)$ to denote $\frac{\partial V_{\mathrm{M}}(e)}{\partial x} F(x)$

and $\frac{\partial V_{\mathrm{M}}(e)}{\partial x} G_r(x)$, respectively. We will also use similar notations in the rest of the paper.

Under (27), Eq. (8) reduces to $\dot{x} = F(x)$.

Let S be the largest invariant subset of $\{e \in \mathbb{R}^{Nn} | \dot{V}_{\mathrm{M}} = 0\}$. Hence, for any $e(0) \in S$, $e(t, x(0))$ stays in S forever, where $e(t, x(0)) = (L \otimes I_n)x(t, x(0))$ and $x(t, x(0))$ is the trajectory of $\dot{x} = F(x)$ starting from $x(0)$. This, together with (26) and (27), imply that, for any $e(0) \in S$,

$$L_F V_{\mathrm{M}}(e(t, x(0))) = 0, \quad t \geq 0 \tag{28}$$

and

$$L_{G_r} V_{\mathrm{M}}(e(t, x(0))) = 0, \quad t \geq 0,\ 1 \leq r \leq Nm. \tag{29}$$

Define $\xi = \mathrm{ad}_F G_r(x(t, x(0)))$, $1 \leq r \leq Nm$. In view of (28) and (29), it follows that

$$\begin{aligned} L_\xi V_{\mathrm{M}}(e(t, x(0))) &= L_F L_{G_r} V_{\mathrm{M}}(e(t, x(0))) \\ &\quad - L_{G_r} L_F V_{\mathrm{M}}(e(t, x(0))) \\ &= 0. \end{aligned} \tag{30}$$

Similar manipulations lead to

$$L_\xi V_{\mathrm{M}}(e(t, x(0))) = 0, \quad \forall\, e(0) \in S, \tag{31}$$

where $\xi = \mathrm{ad}_F^q G_r(x(t, x(0)))$, $q \in \mathbb{N}$, $1 \leq r \leq Nm$. Under Assumption 3, it follows from Lemma 5 that $\dim \mathrm{span}\{ad_F^q G_r(x) : 0 \leq q \leq n-1, 1 \leq r \leq Nm\} = Nn$, $\forall x \in \mathbb{R}^{Nn}$. This, along with (31), imply that

$$\begin{aligned} S &= \left\{ e \in \mathbb{R}^{Nn} \middle| \frac{\partial V_{\mathrm{M}}(e)}{\partial x} = \mathbf{0}_{1 \times Nn} \right\} \\ &= \{ e \in \mathbb{R}^{Nn} | 2x^{\mathrm{T}}(L \otimes P) = \mathbf{0}_{1 \times Nn} \} \\ &= \{ e \in \mathbb{R}^{Nn} | (I_N \otimes P)e = \mathbf{0}_{Nn \times 1} \}. \end{aligned} \tag{32}$$

Since P is nonsingular, $S = \{e = \mathbf{0}_{Nn \times 1}\}$. Therefore, by LaSalle's Invariance Principle, $\lim_{t \to \infty} e(t) = 0$, and consensus is achieved. □

Remark 4. If there is a constraint on the magnitude of the control inputs of the agents, we can modify the control input to each agent i as

$$u_{ik} = \mathrm{sign}(-g_k^{\mathrm{T}}(x_i)Pe_i) \min(|g_k^{\mathrm{T}}(x_i)Pe_i|, \alpha), \tag{33}$$

where $u_{ik}, k = 1, 2, \ldots, m$, is the kth element in u_i, and $\alpha > 0$ is the bound on the inputs. The bounded control inputs (33) then bring the multi-agent system into consensus if Assumptions 1–3 are satisfied.

5. Consensus in the Presence of Communication Delays

In this section, we take communication delays into consideration. In this case, we restrict $g(x_i)$ in (1) to a constant matrix, that is,

$$\dot{x}_i = f(x_i) + Bu_i. \tag{34}$$

We assume the communication delays to be a constant τ. We propose the following distributed low gain control laws,

$$u_i(t) = -\varepsilon B^{\mathrm{T}} P e_i(t - \tau), \tag{35}$$

where ε is a low gain parameter. Furthermore, with the help of the notations in (3)–(7), we have

$$u(t) = -\varepsilon(I_N \otimes B^{\mathrm{T}}P)e(t - \tau). \tag{36}$$

The dynamics of the consensus errors are given by

$$\dot{e}(t) = (L \otimes I_n)\big(F(x(t)) - \varepsilon(I_N \otimes BB^{\mathrm{T}}P)e(t - \tau)\big). \tag{37}$$

Assumption 4. There exists a constant $\beta > 0$ such that,

$$\begin{aligned} &(f(x_i) - f(x_j))^{\mathrm{T}} PBB^{\mathrm{T}} P(f(x_i) - f(x_j)) \\ &\quad \leq \beta(x_i - x_j)^{\mathrm{T}} PBB^{\mathrm{T}} P(x_i - x_j), \quad \forall\, x_i, x_j \in \mathbb{R}^n, \end{aligned}$$

where the matrix P is as defined in Assumption 2.

To establish the result regarding the behavior of the closed-loop multi-agent system driven by the control inputs (35), we need the following technical lemmas.

Lemma 6 ([12]). *Consider a set of real vectors a_i, $i = 1, 2, \ldots, N$. Then,*

$$\left(\sum_{i=1}^N a_i\right)^{\mathrm{T}} \left(\sum_{i=1}^N a_i\right) \leq N \sum_{i=1}^N a_i^{\mathrm{T}} a_i.$$

Lemma 7 ([12]). *Let a and b be real numbers, and $f(s)$ be an integrable real-valued function. Then,*

$$\left(\int_a^b f(s)ds\right)^2 \leq (b - a) \int_a^b f^2(s)ds.$$

Lemma 8 ([13]). *Consider two positive semi-definite matrices A and B. The product of the two matrices, AB, is positive semi-definite if and only if AB is symmetric.*

Lemma 9. *Define a function, in terms of any given integrable function $f(t)$, as follows,*

$$h(t) = \int_0^\tau \int_{t - s_1}^t f(s_2)ds_2 ds_1,$$

where $\tau \in \mathbb{R}^+$ is a constant. Then,

$$\frac{d}{dt}h(t) = \tau f(t) - \int_{t-\tau}^t f(s_2)ds_2.$$

Proof. Switching the order of integration, we can rewrite $h(t)$ as

$$h(t) = \int_{t-\tau}^t \int_{t - s_2}^\tau f(s_2)ds_1 ds_2 = \int_{t-\tau}^t (\tau - t + s_2) f(s_2)ds_2,$$

from which we have, by the Leibniz rule,

$$\begin{aligned}\frac{d}{dt}h(t) &= -\int_{t-\tau}^{t} f(s_2)ds_2 + (\tau - t + t)f(t) \\ &\quad -(\tau - t + t - \tau)f(t-\tau) \\ &= \tau f(t) - \int_{t-\tau}^{t} f(s_2)ds_2.\end{aligned}$$

□

Theorem 2. *Consider a multi-agent system whose agent dynamics are described by* (34). *Suppose that Assumptions* 1–4 *are all satisfied. There exists a constant* $\bar{\tau} > 0$ *such that for any* $\tau \in [0, \bar{\tau}]$, *the control laws* (35) *steer the system into global consensus as long as the low gain parameter* ε *is sufficiently small. That is, for any* $\tau \in [0, \bar{\tau}]$, *there exists an* $\bar{\varepsilon}$ *such that for any* $\varepsilon \in (0, \bar{\varepsilon}]$, $\lim_{t\to\infty} e(t) = 0$ *for all* $x_i(0) \in \mathbb{R}^n$, $i = 1, 2, \ldots, N$.

Proof. We consider the following Lyapunov function,

$$V_{\mathrm{M}}(e) = V_1(e) + V_2(e) + V_3(e) \tag{38}$$

with

$$\begin{aligned}V_1(e) &= e^{\mathrm{T}}(t)(M \otimes P)e(t), \\ V_2(e) &= \varepsilon\tau \int_0^{\tau}\int_{t-s_1}^{t} \dot{e}^{\mathrm{T}}(s_2)\Phi\dot{e}(s_2)ds_2ds_1, \\ V_3(e) &= \left(\sum_{k=0}^{\infty} aq^k\right)\int_0^{\tau}\int_{t-s_1}^{t} \dot{e}^{\mathrm{T}}(s_2)\Psi\dot{e}(s_2)ds_2ds_1,\end{aligned}$$

where $a = 8\varepsilon\tau^3 N\beta$, $q = 4\tau^2\beta$, and $\Phi = I_N \otimes PBB^{\mathrm{T}}P$ and $\Psi = M \otimes PBB^{\mathrm{T}}P$ are positive semi-definite matrices. To guarantee the convergence in V_3, we require $q < 1$, i.e., $\tau < \frac{1}{2\sqrt{\beta}}$. We note that V_1 is the Lyapunov function used in the proof of Theorem 1 for the delay-free case. Because of the presence of communication delays, a positive semi-definite term appears in the time derivative of V_1 and in order to cancel that term we introduce V_2. However, the time derivative of V_2 in turn contains another positive semi-definite term, so we introduce the first term in V_3. Similarly, every time we add a new term to the Lyapunov function to cancel the positive semi-definite term in the derivative resulting from the previously introduced term in the Lyapunov function, the new term itself resulting in a new positive semi-definite term in the derivative. This is the reason why V_3 contains infinitely many terms.

Taking time derivative of V_1 yields

$$\dot{V}_1 = 2e^{\mathrm{T}}(t)(M \otimes P)\dot{e}(t), \tag{39}$$

which, together with (37), give

$$\begin{aligned}\dot{V}_1 &= 2e^{\mathrm{T}}(t)(ML \otimes P)F(x(t)) \\ &\quad - 2\varepsilon e^{\mathrm{T}}(t)(ML \otimes PBB^{\mathrm{T}}P)e(t-\tau) \\ &= 2e^{\mathrm{T}}(t)(I_N \otimes P)F(x(t)) \\ &\quad - 2\varepsilon e^{\mathrm{T}}(t)(I_N \otimes PBB^{\mathrm{T}}P)e(t-\tau).\end{aligned} \tag{40}$$

Note that

$$e(t) = e(t-\tau) + \int_{t-\tau}^{t} \dot{e}(\sigma)d\sigma. \tag{41}$$

Equation (40) can then be continued as

$$\begin{aligned}\dot{V}_1 &= 2e^{\mathrm{T}}(t)(I_N \otimes P)F(x(t)) \\ &\quad - 2\varepsilon e^{\mathrm{T}}(t-\tau)(I_N \otimes PBB^{\mathrm{T}}P)e(t-\tau) \\ &\quad - 2\varepsilon e^{\mathrm{T}}(t-\tau)(I_N \otimes PBB^{\mathrm{T}}P)\int_{t-\tau}^{t} \dot{e}(\sigma)d\sigma \\ &\le 2e^{\mathrm{T}}(t)(I_N \otimes P)F(x(t)) - \varepsilon e^{\mathrm{T}}(t-\tau)\Phi e(t-\tau) \\ &\quad + \varepsilon\int_{t-\tau}^{t} \dot{e}^{\mathrm{T}}(\sigma)d\sigma\Phi\int_{t-\tau}^{t} \dot{e}(\sigma)d\sigma.\end{aligned}$$

By Lemma 7, it follows that

$$\begin{aligned}\dot{V}_1 &\le 2e^{\mathrm{T}}(t)(I_N \otimes P)F(x(t)) - \varepsilon e^{\mathrm{T}}(t-\tau)\Phi e(t-\tau) \\ &\quad + \varepsilon\tau\int_{t-\tau}^{t} \dot{e}^{\mathrm{T}}(\sigma)\Phi\dot{e}(\sigma)d\sigma.\end{aligned} \tag{42}$$

For V_2, by Lemma 9, we have

$$\dot{V}_2 = \varepsilon\tau^2\dot{e}^{\mathrm{T}}(t)\Phi\dot{e}(t) - \varepsilon\tau\int_{t-\tau}^{t} \dot{e}^{\mathrm{T}}(\sigma)\Phi\dot{e}(\sigma)d\sigma. \tag{43}$$

In view of (37), it follows from Lemma 6 that

$$\begin{aligned}\dot{V}_2 &\le 2\varepsilon\tau^2F^{\mathrm{T}}(x(t))(L \otimes I_n)\Phi(L \otimes I_n)F(x(t)) \\ &\quad + 2\varepsilon^3\tau^2e^{\mathrm{T}}(t-\tau)\Phi(L^2 \otimes BB^{\mathrm{T}})\Phi e(t-\tau) \\ &\quad - \varepsilon\tau\int_{t-\tau}^{t} \dot{e}^{\mathrm{T}}(\sigma)\Phi\dot{e}(\sigma)d\sigma.\end{aligned} \tag{44}$$

We expand the first term in the above inequality and obtain

$$\begin{aligned}&F^{\mathrm{T}}(x(t))(L \otimes I_n)\Phi(L \otimes I_n)F(x(t)) \\ &\quad = \sum_{i=1}^{N}\left\|\sum_{j=1}^{N} a_{ij}B^{\mathrm{T}}P(f(x_i) - f(x_j))\right\|^2 \\ &\quad \le N\sum_{i=1}^{N}\sum_{j=1}^{N}\|a_{ij}B^{\mathrm{T}}P(f(x_i) - f(x_j))\|^2,\end{aligned} \tag{45}$$

where we have applied Lemma 6. By Assumption 4, it follows from (45) that

$$\begin{aligned}&F^{\mathrm{T}}(x(t))(L \otimes I_n)\Phi(L \otimes I_n)F(x(t)) \\ &\quad \le N\beta\sum_{i=1}^{N}\sum_{j=1}^{N}\left\|a_{ij}B^{\mathrm{T}}P(x_i - x_j)\right\|^2\end{aligned}$$

$$\begin{aligned} &= 2N\beta x^{\mathrm{T}}(t)(L \otimes PBB^{\mathrm{T}}P)x(t) \\ &= 2N\beta e^{\mathrm{T}}(t)(M \otimes PBB^{\mathrm{T}}P)e(t). \end{aligned} \tag{46}$$

Substituting (41) and (46) into (44), and applying Lemmas 6 and 7, we obtain

$$\begin{aligned} \dot{V}_2 \leq\ & 8\varepsilon\tau^2 N\beta e^{\mathrm{T}}(t-\tau)\Psi e(t-\tau) \\ & + 8\varepsilon\tau^3 N\beta \int_{t-\tau}^{t} \dot{e}^{\mathrm{T}}(\sigma)\Psi\dot{e}(\sigma)d\sigma \\ & + 2\varepsilon^3\tau^2 e^{\mathrm{T}}(t-\tau)\Phi(L^2 \otimes BB^{\mathrm{T}})\Phi e(t-\tau) \\ & - \varepsilon\tau \int_{t-\tau}^{t} \dot{e}^{\mathrm{T}}(\sigma)\Phi\dot{e}(\sigma)d\sigma. \end{aligned} \tag{47}$$

Expanding the third term in the right-hand side of (47) to a summation form, as what we did in (45), and applying Lemmas 3 and 6, we have

$$\begin{aligned} \dot{V}_2 \leq\ & 8\varepsilon\tau^2 N\beta e^{\mathrm{T}}(t-\tau)\Psi e(t-\tau) \\ & + 8\varepsilon\tau^3 N\beta \int_{t-\tau}^{t} \dot{e}^{\mathrm{T}}(\sigma)\Psi\dot{e}(\sigma)d\sigma \\ & + 4\varepsilon^3\tau^2 N e^{\mathrm{T}}(t-\tau)\Phi(L \otimes BB^{\mathrm{T}})\Phi e(t-\tau) \\ & - \varepsilon\tau \int_{t-\tau}^{t} \dot{e}^{\mathrm{T}}(\sigma)\Phi\dot{e}(\sigma)d\sigma. \end{aligned} \tag{48}$$

By Lemma 9, the time derivative of V_3 is given by

$$\begin{aligned} \dot{V}_3 =\ & \sum_{k=0}^{\infty} aq^k\tau\dot{e}^{\mathrm{T}}(t)\Psi\dot{e}(t) \\ & - \sum_{k=0}^{\infty} aq^k \int_{t-\tau}^{t} \dot{e}^{\mathrm{T}}(\sigma)\Psi\dot{e}(\sigma)d\sigma. \end{aligned} \tag{49}$$

Again, recalling (37), it follows from Lemma 6 that

$$\begin{aligned} \dot{V}_3 \leq\ & \sum_{k=0}^{\infty} 2aq^k\tau F^{\mathrm{T}}(x(t))(L \otimes PBB^{\mathrm{T}}P)F(x(t)) \\ & + \sum_{k=0}^{\infty} 2aq^k\varepsilon^2\tau e^{\mathrm{T}}(t-\tau)\Phi(L \otimes BB^{\mathrm{T}})\Phi e(t-\tau) \\ & - \sum_{k=0}^{\infty} aq^k \int_{t-\tau}^{t} \dot{e}^{\mathrm{T}}(\sigma)\Psi\dot{e}(\sigma)d\sigma. \end{aligned} \tag{50}$$

By Assumption 4, (50) can be continued as

$$\begin{aligned} \dot{V}_3 \leq\ & \sum_{k=0}^{\infty} 2aq^k\tau\beta x^{\mathrm{T}}(t)(L \otimes PBB^{\mathrm{T}}P)x(t) \\ & + \sum_{k=0}^{\infty} 2aq^k\varepsilon^2\tau e^{\mathrm{T}}(t-\tau)\Phi(L \otimes BB^{\mathrm{T}})\Phi e(t-\tau) \\ & - \sum_{k=0}^{\infty} aq^k \int_{t-\tau}^{t} \dot{e}^{\mathrm{T}}(\sigma)\Psi\dot{e}(\sigma)d\sigma \\ =\ & \sum_{k=0}^{\infty} 2aq^k\tau\beta e^{\mathrm{T}}(t)\Psi e(t) \\ & + \sum_{k=0}^{\infty} 2aq^k\varepsilon^2\tau e^{\mathrm{T}}(t-\tau)\Phi(L \otimes BB^{\mathrm{T}})\Phi e(t-\tau) \\ & - \sum_{k=0}^{\infty} aq^k \int_{t-\tau}^{t} \dot{e}^{\mathrm{T}}(\sigma)\Psi\dot{e}(\sigma)d\sigma. \end{aligned} \tag{51}$$

Substituting (41) into (51) and applying Lemmas 6 and 7, we have

$$\begin{aligned} \dot{V}_3 \leq\ & \sum_{k=0}^{\infty} 4aq^k\tau\beta e^{\mathrm{T}}(t-\tau)\Psi e(t-\tau) \\ & + \sum_{k=0}^{\infty} 4aq^k\tau^2\beta \int_{t-\tau}^{t} \dot{e}^{\mathrm{T}}(\sigma)\Psi\dot{e}(\sigma)d\sigma \\ & + \sum_{k=0}^{\infty} 2aq^k\varepsilon^2\tau e^{\mathrm{T}}(t-\tau)\Phi(L \otimes BB^{\mathrm{T}})\Phi e(t-\tau) \\ & - \sum_{k=0}^{\infty} aq^k \int_{t-\tau}^{t} \dot{e}^{\mathrm{T}}(\sigma)\Psi\dot{e}(\sigma)d\sigma, \end{aligned} \tag{52}$$

which, by collection of similar terms, simplifies to

$$\begin{aligned} \dot{V}_3 \leq\ & \sum_{k=0}^{\infty} 4aq^k\tau\beta e^{\mathrm{T}}(t-\tau)\Psi e(t-\tau) \\ & + \sum_{k=0}^{\infty} 2aq^k\varepsilon^2\tau e^{\mathrm{T}}(t-\tau)\Phi(L \otimes BB^{\mathrm{T}})\Phi e(t-\tau) \\ & - 8\varepsilon\tau^3 N\beta \int_{t-\tau}^{t} \dot{e}^{\mathrm{T}}(\sigma)\Psi\dot{e}(\sigma)d\sigma. \end{aligned} \tag{53}$$

Summing both sides of (42), (48), and (53) yields

$$\begin{aligned} \dot{V}_{\mathrm{M}} \leq\ & 2e^{\mathrm{T}}(t)(I_N \otimes P)F(x(t)) - \varepsilon e^{\mathrm{T}}(t-\tau)\Phi e(t-\tau) \\ & + 8\varepsilon\tau^2 N\beta e^{\mathrm{T}}(t-\tau)\Psi e(t-\tau) \\ & + 4\varepsilon^3\tau^2 N e^{\mathrm{T}}(t-\tau)\Phi(L \otimes BB^{\mathrm{T}})\Phi e(t-\tau) \\ & + \sum_{k=0}^{\infty} 4aq^k\tau\beta e^{\mathrm{T}}(t-\tau)\Psi e(t-\tau) \\ & + \sum_{k=0}^{\infty} 2aq^k\varepsilon^2\tau e^{\mathrm{T}}(t-\tau)\Phi(L \otimes BB^{\mathrm{T}})\Phi e(t-\tau) \\ =\ & 2e^{\mathrm{T}}(t)(I_N \otimes P)F(x(t)) - \varepsilon e^{\mathrm{T}}(t-\tau)\Phi e(t-\tau) \\ & + \sum_{k=0}^{\infty} 8\varepsilon\tau^2 N\beta q^k e^{\mathrm{T}}(t-\tau)\Psi e(t-\tau) \\ & + \sum_{k=0}^{\infty} 4\varepsilon^3\tau^2 N q^k e^{\mathrm{T}}(t-\tau) \times \Phi(L \otimes BB^{\mathrm{T}})\Phi e(t-\tau). \end{aligned} \tag{54}$$

As mentioned before, when $q < 1$, i.e., $\tau < \frac{1}{2\sqrt{\beta}}$, the summations of geometric series in (54) are finite and therefore (54) can be rewritten as

$$\begin{aligned}\dot{V}_{\mathrm{M}} \le\ & 2e^{\mathrm{T}}(t)(I_N \otimes P)F(x(t)) - \varepsilon e^{\mathrm{T}}(t-\tau)\Phi e(t-\tau) \\ & + \frac{4\varepsilon^3\tau^2 N}{1-4\tau^2\beta} e^{\mathrm{T}}(t-\tau)\Phi(L \otimes BB^{\mathrm{T}})\Phi e(t-\tau) \\ & + \frac{8\varepsilon\tau^2 N\beta}{1-4\tau^2\beta} e^{\mathrm{T}}(t-\tau)(M \otimes I_n)\Phi e(t-\tau). \end{aligned} \tag{55}$$

Since B, L, and M are known and constant, there exist positive constants η_1 and η_2 such that,

$$L \otimes BB^{\mathrm{T}} \le \eta_1 I_{Nn}, \tag{56}$$

$$M \otimes I_n \le \eta_2 I_{Nn}. \tag{57}$$

Substituting (56) and (57) into (55), we have

$$\begin{aligned}\dot{V}_{\mathrm{M}} \le\ & 2e^{\mathrm{T}}(t)(I_N \otimes P)F(x(t)) - \varepsilon e^{\mathrm{T}}(t-\tau)\Phi e(t-\tau) \\ & + \frac{4\varepsilon^3\tau^2 N\eta_1}{1-4\tau^2\beta} e^{\mathrm{T}}(t-\tau)\Phi^2 e(t-\tau) \\ & + \frac{8\varepsilon\tau^2 N\beta\eta_2}{1-4\tau^2\beta} e^{\mathrm{T}}(t-\tau)\Phi e(t-\tau). \end{aligned} \tag{58}$$

We have proved that $2e^{\mathrm{T}}(t)(I_N \otimes P)F(x(t)) \le 0$ in Lemma 4. It is straightforward to obtain from (58) that $\dot{V}_{\mathrm{M}} \le 0$ when

$$\left(\varepsilon - \frac{8\varepsilon\tau^2 N\beta\eta_2}{1-4\tau^2\beta}\right)\Phi - \frac{4\varepsilon^3\tau^2 N\eta_1}{1-4\tau^2\beta}\Phi^2 \ge 0. \tag{59}$$

Since $\Phi \ge 0$ and the left side of (59) is symmetric, by Lemma 8, (59) is true when

$$\left(\varepsilon - \frac{8\varepsilon\tau^2 N\beta\eta_2}{1-4\tau^2\beta}\right)I_{Nn} - \frac{4\varepsilon^3\tau^2 N\eta_1}{1-4\tau^2\beta}\Phi \ge 0. \tag{60}$$

In order to satisfy (60), we need the following two conditions

$$\varepsilon - \frac{8\varepsilon\tau^2 N\beta\eta_2}{1-4\tau^2\beta} > 0 \tag{61}$$

and

$$\varepsilon - \frac{8\varepsilon\tau^2 N\beta\eta_2}{1-4\tau^2\beta} - \frac{4\varepsilon^3\tau^2 N\eta_1}{1-4\tau^2\beta}\lambda_{\max}(\Phi) > 0, \tag{62}$$

where $\lambda_{\max}(\Phi)$ is the maximum eigenvalue of the positive semi-definite matrix $\Phi = I_N \otimes PBB^{\mathrm{T}}P$. According to the preceding derivations, $\dot{V}_{\mathrm{M}} \le 0$ when both (61) and (62) are satisfied. Then, solving (61) and (62), we obtain $\tau \le \bar{\tau}$ and $\varepsilon \le \bar{\varepsilon}$, where $\bar{\tau}$ and $\bar{\varepsilon}$ are such that

$$\bar{\tau} < \sqrt{\frac{1}{4\beta(2N\eta_2+1)}},$$

$$\bar{\varepsilon} < \sqrt{\frac{1-4\tau^2\beta(2N\eta_2+1)}{4\tau^2 N\eta_1\lambda_{\max}(\Phi)}}.$$

Clearly, $\bar{\tau} < \frac{1}{2\sqrt{\beta}}$.

From (58), we observe that $\dot{V}_{\mathrm{M}} = 0$ if and only if

$$2e^{\mathrm{T}}(t)(I_N \otimes P)F(x(t)) = 0 \tag{63}$$

and

$$\begin{aligned} e^{\mathrm{T}}(t-\tau)\Bigg(\left(\varepsilon - \frac{8\varepsilon\tau^2 N\beta\eta_2}{1-4\tau^2\beta}\right)\Phi \\ - \frac{4\varepsilon^3\tau^2 N\eta_1}{1-4\tau^2\beta}\Phi^2\Bigg)e(t-\tau) = 0, \end{aligned} \tag{64}$$

which is equivalent to

$$e^{\mathrm{T}}(t-\tau)(I_N \otimes PB)\Gamma(I_N \otimes B^{\mathrm{T}}P)e(t-\tau) = 0, \tag{65}$$

where

$$\begin{aligned}\Gamma = & \left(\varepsilon - \frac{8\varepsilon\tau^2 N\beta\eta_2}{1-4\tau^2\beta}\right)I_{Nm} \\ & - \frac{4\varepsilon^3\tau^2 N\eta_1}{1-4\tau^2\beta}(I_N \otimes B^{\mathrm{T}}PPB). \end{aligned} \tag{66}$$

Since Φ and $(I_N \otimes B^{\mathrm{T}}PPB)$ have the same nonzero eigenvalues, under (61) and (62), we have $\Gamma > 0$. Then, (65) is true if and only if

$$e^{\mathrm{T}}(t-\tau)(I_N \otimes PB) = \mathbf{0}_{1\times Nm}. \tag{67}$$

In view of (63) and (67), following similar derivations as in the proof of Theorem 1, we conclude that the largest invariant set in which $\dot{V}_{\mathrm{M}} = 0$ is $S = \{e = \mathbf{0}_{Nn\times 1}\}$. Therefore, by LaSalle's Invariance Principle, $\lim_{t\to\infty} e(t) = 0$. This completes the proof. □

6. Simulation Results

In this section, the performance of the proposed control laws is demonstrated by numerical simulation. In all our simulation, we consider multi-agent systems with five agents and a communication topology shown in Fig. 1, which is an undirected and connected graph.

A. *Example* 1: *Linear unforced dynamics without delays*

The dynamics of agent i, $i = 1, 2, \ldots, 5$ are given as follows

$$\dot{x}_i = Ax_i + g(x_i)u_i, \tag{68}$$

where $x_i \in \mathbb{R}^2$, $u_i \in \mathbb{R}$,

$$A = \begin{bmatrix} -1.5 & 3 \\ 1 & -2 \end{bmatrix}, \quad g(x_i) = \begin{bmatrix} 0 \\ 1 + x_{i1}^2 + x_{i2}^2 \end{bmatrix}.$$

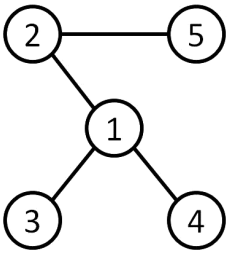

Fig. 1. Communication topology.

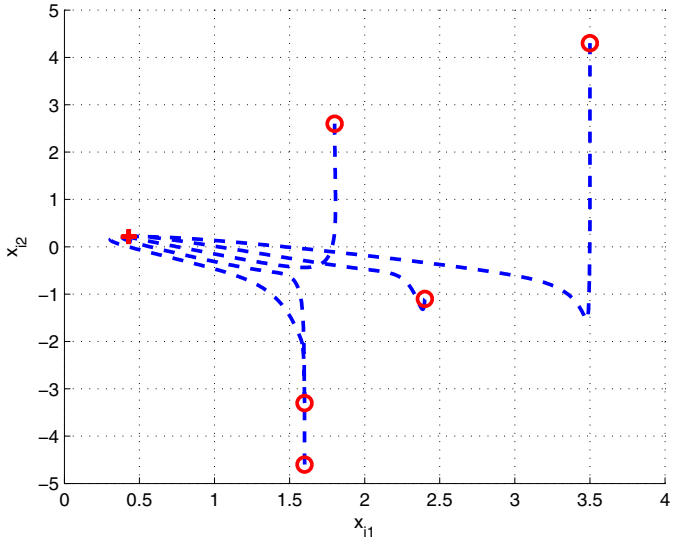

Fig. 2. Example 1: Trajectories of agents, $\tau = 0$.

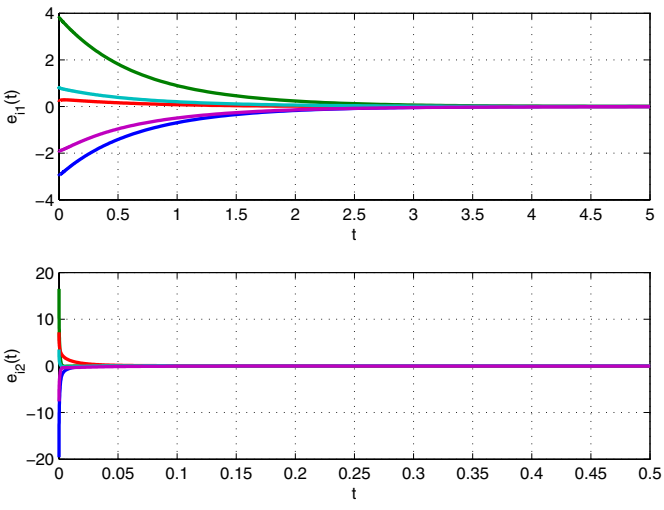

Fig. 3. Example 1: Consensus errors, $\tau = 0$.

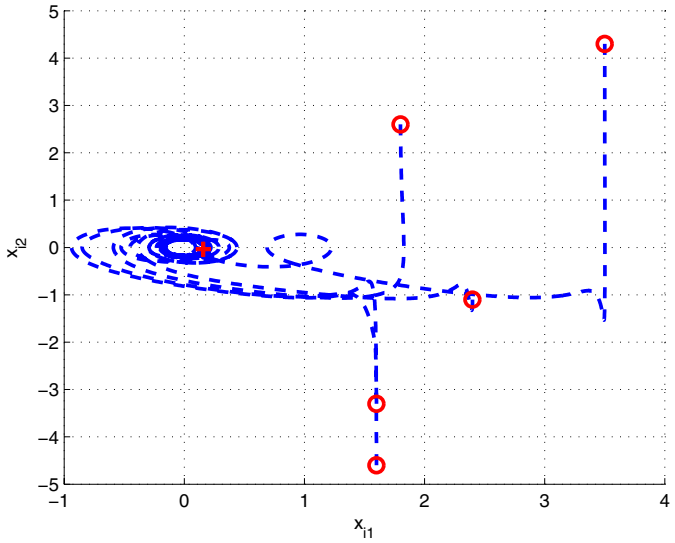

Fig. 4. Example 2: Trajectories of agents, $\tau = 0$.

It can be easily verified that Assumption 3 holds and Assumption 2 is satisfied with

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

Shown in Figs. 2 and 3 are trajectories of the agents and the consensus errors. In the simulation, the initial states of the agents are randomly chosen as $x_1(0) = [1.6, -4.6]^{\mathrm{T}}$, $x_2(0) = [3.5, 4.3]^{\mathrm{T}}$, $x_3(0) = [1.8, 2.6]^{\mathrm{T}}$, $x_4(0) = [2.4, -1.1]^{\mathrm{T}}$, and $x_5(0) = [1.6, -3.3]^{\mathrm{T}}$. As observed in Fig. 2, where circles indicate the initial states and crosses represent the states when simulation terminates, the sates of the agents are synchronized. We can see more clearly in Fig. 3 that consensus errors converge to zero as the system evolves.

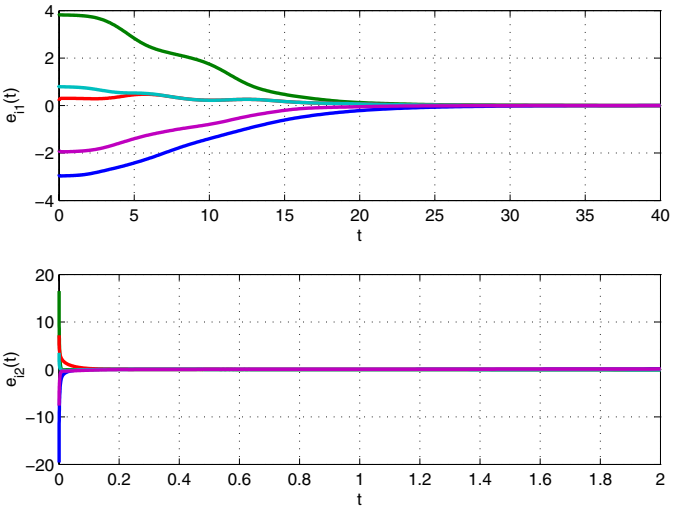

Fig. 5. Example 2: Consensus errors, $\tau = 0$.

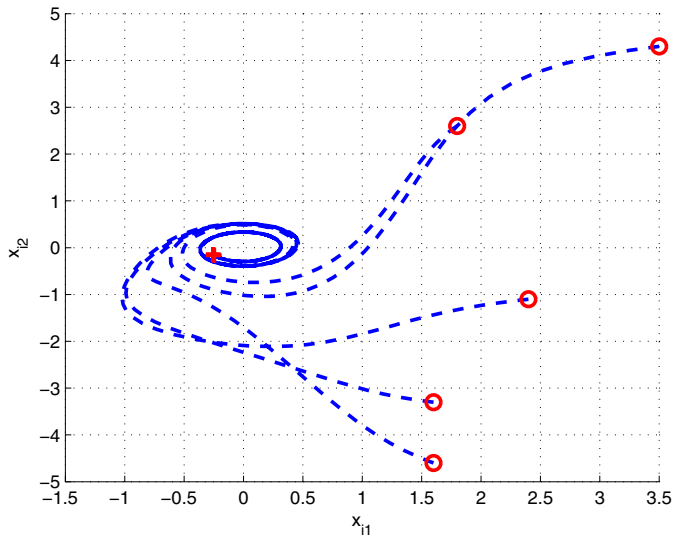

Fig. 6. Example 3: Trajectories of agents, $\tau = 0.1$, $\varepsilon = 0.5$.

B. *Example 2: Nonlinear unforced dynamics without delays*

The dynamics of agent i, $i = 1, 2, \ldots, 5$, are given by,

$$\begin{aligned} \dot{x}_{i1} &= x_{i2}, \\ \dot{x}_{i2} &= -x_{i1} - x_{i2}^3 + (1 + x_{i1}^2 + x_{i2}^2)u_i. \end{aligned} \tag{69}$$

It can be easily verified that both Assumptions 2 and 3 are satisfied, with $P = I_2$.

Trajectories of the agents under the influence of the control laws (12) are shown in Fig. 4, where circles and crosses respectively indicate the initial states and the states when simulation terminates. The evolution of the errors are shown in Fig. 5. In the simulation, we have chosen the initial conditions to be $x_1(0) = [1.6, -4.6]^{\mathrm{T}}$, $x_2(0) = [3.5, 4.3]^{\mathrm{T}}$,

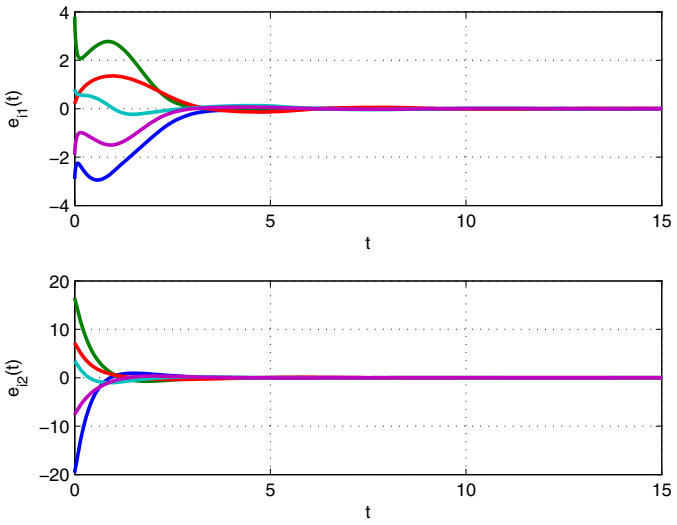

Fig. 7. Example 3: Consensus errors, $\tau = 0.1$, $\varepsilon = 0.5$.

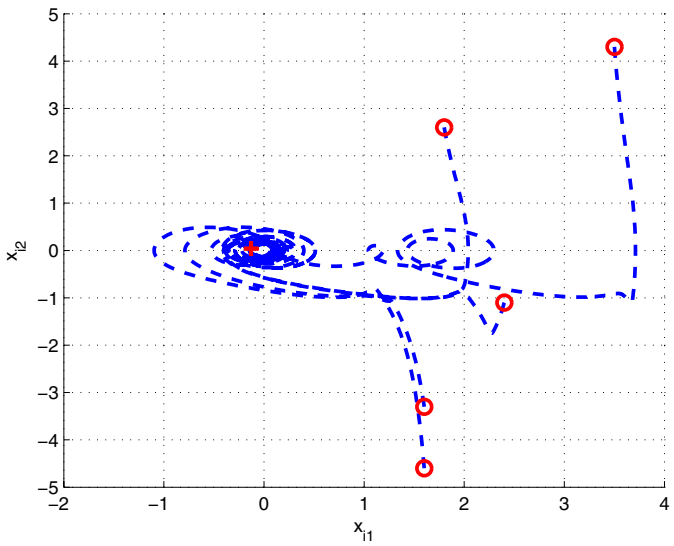

Fig. 8. Example 4: Trajectories of agents, $\tau = 0$.

$x_3(0) = [1.8, 2.6]^{\mathrm{T}}$, $x_4(0) = [2.4, -1.1]^{\mathrm{T}}$, and $x_5(0) = [1.6, -3.3]^{\mathrm{T}}$. Again, the states of all the agents reach an agreement and errors approach zero.

C. *Example 3: Nonlinear unforced dynamics with delays*

The dynamics of agent i, $i = 1, 2, \ldots, 5$, are given by,

$$\begin{aligned} \dot{x}_{i1} &= -x_{i1}^3 + x_{i2}, \\ \dot{x}_{i2} &= -x_{i1} + u_i, \end{aligned} \tag{70}$$

which satisfy all the proposed assumptions, with $P = I_2$ and $\beta = 1$. The communication delay is $\tau = 0.1$. The low gain parameter is thus chosen to be $\varepsilon = 0.5$.

We have chosen the initial conditions to be $x_1(t) = [1.6, -4.6]^{\mathrm{T}}$, $x_2(t) = [3.5, 4.3]^{\mathrm{T}}$, $x_3(t) = [1.8, 2.6]^{\mathrm{T}}$,

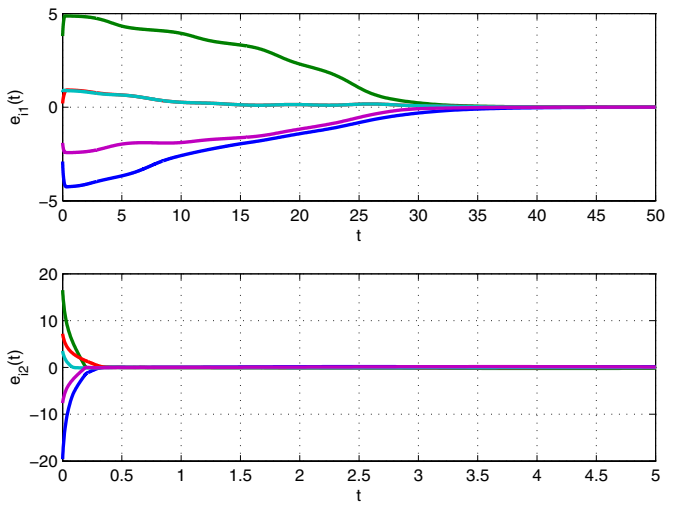

Fig. 9. Example 4: Consensus errors e_{i1}, $\tau = 0$.

$x_4(t) = [2.4, -1.1]^{\mathrm{T}}$, and $x_5(t) = [1.6, -3.3]^{\mathrm{T}}$, for $t \in [-\tau, 0]$. Simulation results are shown in Figs. 6 and 7, from which consensus is observed.

D. *Example 4: Consensus under actuator saturation*

Consider the multi-agent system described by (69) and assume that the magnitude of the control input of each agent is limited by 1. Following Remark 4, the appropriately modified bounded control laws, with $\alpha = 1$, would ensure consensus. To carry out simulation, we choose the initial states of the agents as $x_1(0) = [1.6, -4.6]^{\mathrm{T}}$, $x_2(0) = [3.5, 4.3]^{\mathrm{T}}$, $x_3(0) = [1.8, 2.6]^{\mathrm{T}}$, $x_4(0) = [2.4, -1.1]^{\mathrm{T}}$, and $x_5(0) = [1.6, -3.3]^{\mathrm{T}}$. The simulation results in Figs. 8 and 9 show that the consensus errors converge to zero and the states of agents are synchronized.

7. Conclusions

Consensus behavior of multi-agent systems with control-affine nonlinear dynamics was studied in this paper. We proposed distributed control laws based on neighborhood consensus errors that achieve global consensus. In this work, the communication topology is assumed to be undirected. Future work will focus on more general communication topologies.

Acknowledgment

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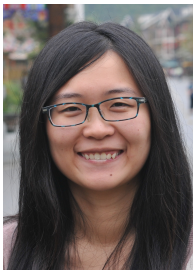

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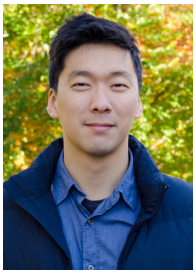

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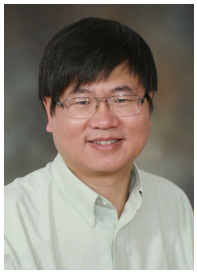

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