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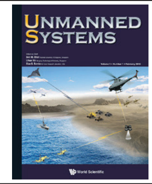

Properties of Eigenvalue/Eigenvector of Hypermatrices

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Three properties of the universal (U-) eigenvalues/eigenvectors of hypermatrices are proposed and proved. (1) The eigenvalue/eigenvector of Kronecker product of hypermatrices is the product of eigenvalues/eigenvectors of factor hypermatrices. (2) The similarity of hypermatrices is defined, and then it is proved that the similarity ensures the same eigenvalues/eigenvectors. (3) The Cayley–Hamilton Theorems of Hypermatrix. These properties justify the reasonability of U-engenvalues/eigenvectors.

Keywords: Hypermatrix; Hypervector; U- (D-) eigenproblem; *t*-product; semi-tensor product of matrices.

1. Introduction

The eigenvalue/eigenvector of matrices plays a fundamental role in the matrix theory and has many applications. Particularly in unmanned systems ranging from single-vehicle autonomous navigation to multi-agent swarm coordination, eigenvalue/eigenvector analysis provides the geometric foundation for state-space dynamics. To reveal the geometric meaning of eigenvalue/eigenvector, we consider a matrix as a linear mapping over a fixed dimensional Euclidian space, say, $\mathbb{R}^n$. Then geometrically an eigenvector represents a one-dimensional invariant subspace w.r.t. the mapping determined by the given matrix. While, the corresponding eigenvalue represents the proportional rate of the mapping.

The classical eigenvalue/eigenvector is only defined for square matrices. A generalized definition is described by the following eigenequation:

$$(A-\lambda B)x=0, \tag{1}$$

where B is called the type of the eigenvalue/eigenvector. When $B=I_n$ (the identity matrix), the solution $\{\lambda, x\neq 0\}$ becomes the standard eigenvalue/eigenvector. When $B\neq I_n$, then the generalized eigenvector does not represent an invariant subspace, but a subspace with certain relationship with the subspace determined by x. The relationship is determined by B.

A more general case is that the matrix $A\in\mathcal{M}_{m\times n}$ is not a square matrix [1]. Then A determines a mapping $\mathbb{R}^n\to\mathbb{R}^m$, and the solution $\{\lambda, x\neq 0\}$ of (1) has similar meaning as in square case.

Motivated by the importance of eigenvalue/eigenvector in matrix analysis and applications, extending the notions of eigenvalues/eigenvectors to the hypermatrix setting becomes natural. In 2005, Qi [2] and Lim [3] first introduced the definition of tensor eigenvalues/eigenvectors independently. Since then, studies have been devoted to the eigenvalue problems of hypermatrices (they are also commonly called tensors). In unmanned systems, hypermatrices naturally arise in multi-sensor data fusion, swarm formation control, and higher-order signal processing tasks. Unmanned aerial vehicles (UAVs) and autonomous ground vehicles generate inherently multi-way data through distributed sensor arrays, hyper-spectral imaging, and LiDAR point clouds, necessitating efficient eigen-decomposition methods for real-time onboard processing.

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Let $\mathcal{A} = (a_{i_1,\dots,i_d}) \in \mathbb{R}^{\overbrace{n\times\cdots\times n}^{d}}$ be a hypermatrix with order d and dimension $\underbrace{n\times\cdots\times n}_{d}$ [4]. Arranging the elements of $\mathcal{A}$ into a matrix as

$$M_A := \begin{bmatrix} a_{1,1,\dots,1} & a_{1,1,\dots,2} & \cdots & a_{1,n,\dots,n} \\ a_{2,1,\dots,1} & a_{2,1,\dots,2} & \cdots & a_{2,n,\dots,n} \\ \vdots & & & \\ a_{n,1,\dots,1} & a_{n,1,\dots,2} & \cdots & a_{n,n,\dots,n} \end{bmatrix}, \tag{2}$$

some existing definitions of eigenvalue/eigenvector for hypermatrices are as follows.

(i) The H-eigenvalue/eigenvector [2]:

$$M_A x^{d-1} = \lambda x^{[d-1]}, \quad x^{[d-1]} := (x_1^{d-1},\dots,x_n^{d-1})^{\mathrm{T}}. \tag{3}$$

(ii) The E- (or Z-) eigenvalue/eigenvector [2]:

$$M_A x^{d-1} = \lambda x \quad \text{with } x^{\mathrm{T}}x = 1. \tag{4}$$

(iii) The D-eigenvalue/eigenvector [5]:

$$M_A x^{d-1} = \lambda D x \quad \text{with } x^{\mathrm{T}}Dx = 1. \tag{5}$$

(iv) The Markov chain-based eigenvalue/eigenvector [6, 7]:

$$M_A x^{d-1} = \lambda (x_1 + \cdots + x_n)^{d-2} x. \tag{6}$$

(v) The generalized eigenvalue/eigenvector [8]:

$$M_A x^{d-1} = \lambda B x^{d-1}. \tag{7}$$

(vi) The cubic eigenvalue/eigenvector with inner product [9]:

$$M_A x^3 = \lambda (x^{\mathrm{T}}x) x. \tag{8}$$

The aforementioned eigenvalues/eigenvectors of hypermatrices have revealed some useful properties of hypermatrices, and have received many applications. However, they are all restricted on "vector form" of eigenvectors. In certain sense, they are only interested in one-dimensional subspaces.

Recently, the U-eigenvalue/eigenvector of hypermatrices is proposed in [10]. The immediate motivation of this kind of eigenvalue/eigenvector is based on the following observation: Unlike a matrix, which can only be considered as a linear mapping from one-dimensional subspace to another identical or different one-dimensional subspace, a hypermatrix of order d can be considered as a multi-linear mapping from $\mathbb{R}^{\overbrace{n\times\cdots\times n}^{r}}$ (or $\mathbb{R}^{rn}$) to $\mathbb{R}^{\overbrace{n\times\cdots\times n}^{s}}$ (or $\mathbb{R}^{sn}$), where $r, s \geq 1$ and $r + s = d$. Hence, instead of "vector form" of eigenvector, [10] proposes a "hypervector form" eigenvector. A hypervector of order r can uniquely represent a subspace of $\mathbb{R}^n$ spanned by its component vectors $x = \ltimes_{i=1}^{r} x_i \sim \mathrm{Span}(x_1,\dots,x_r)$. Hence if a hypermatrix maps an r-dimensional subspace to an s-dimensional subspace, it can be expressed as the hypermatrix maps an hypervector of order r to a hypervector of order s. That is the geometric meaning of U-eigenvalue/eigenvector of hypermatrix. As all the components of the hypervector are the same, it is called a diagonal (D-) hypermatrix. It is clear that most of existing definitions of eigenvector are of this type. Hence, most of the existing kinds of eigenvectors are D-eigenvectors.

To see the U-eigenvalue/eigenvector is more reasonable than the "vector form" eigenvalue/eigenvector, one can see that the U-eigenvalue/eigenvector have many similar properties as the eigenvalue/eigenvector of matrices, such as the linearity that if x is an eigenvector w.r.t. eigenvalue λ, then kx, $k \neq 0$ is also an eigenvector w.r.t. the same λ, etc. Moreover, this paper reveals three properties of U-eigenvalue/eigenvector, which are natural extensions of their counterparts in matrix case. They are: (1) The eigenvalue/eigenvector of Kronecker product of hypermatrices is the product of eigenvalues/eigenvectors of factor hypermatrices. (2) The similar hypermatrices have the same eigenvalues/eigenvectors. (3) Under *t*-product and DK-STP the Cayley–Hamilton Theorem of Hypermatrix can be obtained. These properties justify the definition of U-eigenvalue/eigenvector.

This paper is arranged as follows. Section 2 consists of some preliminaries, including (i) semi-tensor product (STP) of matrices; (ii) matrix expression of hypermatrices; (iii) definition and properties of hypervectors; (iv) U-eigenvalue/eigenvector of hypermatrix. Section 3 considers the Kronecker product of hypermatrices. It is proved that the eigenvalues/eigenvectors of the product of hypermatrices are the product of eigenvalues/eigenvectors of factor hypermatrices. The similarity of hypermatrices is defined in Sec. 4. It is also proved that the similar hypermatrices have same eigenvalues/eigenvectors. Section 5 considers the generalized Cayley–Hamilton Theorem. Based on eigenequation of square matrices, the *t*-product based and the dimension-keeping (DK-) STP based Cayley–Hamilton Theorems are obtained. Then based U-eigenequation, the corresponding Cayley–Hamilton Theorem is also obtained, which shows the reasonability of U-eigenvalue/eigenvector again.

Before ending this introduction, a list of notations is presented as follows:

(1) $\mathbb{R}^n$: n-dimensional real Euclidean space.
(2) $\mathcal{M}_{m\times n}$: the set of $m \times n$ matrices.
(3) $\mathrm{lcm}(a,b)$: least common multiple of a and b.
(4) $\mathbb{R}^{n_1\times n_2\times\cdots\times n_d}$: the dth order hypermatrices of dimensions $n_1 \times \cdots \times n_d$.
(5) $\mathbb{R}^{n_1\ltimes n_2\ltimes\cdots\ltimes n_d}$: the dth order hypervectors of dimensions $n_1, n_2, \dots, n_d$.
(6) $\mathcal{A}, \mathcal{B}, \dots$: hypermatrices.
(7) $\mathbf{i} = \{i_1,\dots,i_d\}$: set of indexes.
(8) $M^{\mathbf{j}\times\mathbf{k}}(\mathcal{A})$: matrix expression of $\mathcal{A}$, which has rows labeled by **j** and columns labeled by **k**.
(9) $[a,b]$: the set of integers $a \leq i \leq b$.
(10) δ_n^i: the ith column of the identity matrix I_n.

(11) $\delta_n^I := [(\delta_n^1)^{\mathrm{T}}, (\delta_n^2)^{\mathrm{T}}, \dots, (\delta_n^n)^{\mathrm{T}}]^{\mathrm{T}}$.
(12) $\Delta_n := \{\delta_n^i | i = 1, \dots, n\}$.
(13) $\delta_n[i_1, \dots, i_s] := [\delta_n^{i_1}, \dots, \delta_n^{i_s}]$.
(14) $\mathbf{1}_\ell := (\underbrace{1, 1, \dots, 1}_{\ell})^{\mathrm{T}}$.
(15) $S(x)$: the subspace spanned by $\{x_1, \dots, x_r\}$, where $x = \ltimes_{i=1}^r x_i$.
(16) $\otimes$: Kronecker product of matrices.
(17) $\ltimes$: semi-tensor product (STP) of matrices.
(18) $\bar{\times}$: dimension keeping semi-tensor product (DK-STP) of matrices.
(19) $\times_{\mathbf{j}}$: contract product of two hypermatrices with respect to their common index subset **j**.
(20) $\mathcal{T}_s^r(V)$: the set of tensors over V with covariant order r and contra-variant order s. When $s = 0$, $\mathcal{T}^r(V)$ denotes the set of covariant tensor over V with covariant order r.
(21) $\boxplus(A) = b\mathrm{circ}(A)$.
(22) $\circledast$: *t*-product.

2. Preliminaries

This section provides some necessary mathematical preliminaries concerning the STP, hypermatrices and hypervectors.

2.1. *STP of matrices*

Definition 2.1 ([11, 12]). Let $A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{p\times q}$, and $t = \mathrm{lcm}(n, p)$. The STP of A and B is defined as

$$A \ltimes B = (A \otimes I_{t/n})(B \otimes I_{t/p}). \tag{9}$$

The STP is a generalization of the classical matrix product, i.e. when $n = p$, $A \ltimes B = AB$. Due to this, we omit the symbol $\ltimes$ in most cases. Some fundamental properties of STP are summarized as follows.

Proposition 2.2. (i) (*Associativity*) *Let A, B, C be three matrices of arbitrary dimensions. Then $(A \ltimes B) \ltimes C = A \ltimes (B \ltimes C)$.*
(ii) (*Distributivity*) *Let A, B be two matrices of same dimension, C is of arbitrary dimension. Then $(A + B) \ltimes C = A \ltimes C + B \ltimes C$, $C \ltimes (A + B) = C \ltimes A + C \ltimes B$.*
(iii) *Let A, B be two matrices of arbitrary dimensions. Then $(A \ltimes B)^{\mathrm{T}} = B^{\mathrm{T}} \ltimes A^{\mathrm{T}}$. If A, B are invertible, then $(A \ltimes B)^{-1} = B^{-1} \ltimes A^{-1}$.*

2.2. *Hypermatrices*

Definition 2.3. (i) A hypermatrix of order d and dimension $n_1 \times \cdots \times n_d$ is a set of data, which are labeled by d indexes i_s, $s \in [1, d]$ with $i_s \in [1, n_s]$. That is,

$$\mathcal{A} = (a_{i_1,\dots,i_d} \mid i_s \in [1, n_s], \quad s \in [1, d]) \in \mathbb{R}^{n_1\times\cdots\times n_d}.$$

(ii) Assume $\mathbf{i} = \{i_1, \dots, i_d\}$, $\mathbf{j} = \{j_1, \dots, j_r\}$, $\mathbf{k} = \{k_1, \dots, k_s\}$, and $\mathbf{i} = \mathbf{j} \cup \mathbf{k}$ be a partition, where $r + s = d$. Then $\mathcal{A}$ can be arranged into a matrix as

$$A = M^{\mathbf{j}\times\mathbf{k}}(\mathcal{A}), \tag{10}$$

where $A = (a_{\alpha,\beta})$ of dimension $n_{\mathbf{k}} \times n_{\mathbf{j}}$ with

$$n_{\mathbf{j}} = \prod_{t=1}^r n_{j_t}, \quad n_{\mathbf{k}} = \prod_{t=1}^s n_{k_t},$$

while its row labels α are indexed by $\{j_1, \dots, j_r\}$, its column labels β are indexed by $\{k_1, \dots, k_s\}$, and the elements are arranged in alphabetic order.

Remark 2.4. (i) If $\mathbf{k} = \emptyset$, then $\mathcal{A}$ is arranged into a column vector as

$$V(\mathcal{A}) = M^{\mathbf{i}\times\emptyset}.$$

(ii) Recall the M_A defined by (2), it is clear that

$$M_A = M^{\{i_1\}\times\{i_2,\dots,i_d\}}(\mathcal{A}).$$

(iii) Note that a hypermatrix of order 2, say, $\mathcal{A} = (a_{i,j})$, is commonly considered as a matrix. In fact, it is not a matrix. Only when its matrix expression is fixed, it becomes either a vector (row or column) or a matrix (A or A^{T}). So the properties of a matrix can be obtained only for a specified matrix expression of $\mathcal{A}$. For a hypermatrix of order $d \geq 3$, it has more matrix expressions. To get its matrix-type of properties, its matrix expression should be fixed in advance. Looking for some matrix-type of properties of a general hypermatrix without fixed matrix expression is meaningless.

2.3. *Hypervectors*

Definition 2.5. (i) Let $x \in \mathbb{R}^n$, where $n = \prod_{i=1}^r n_i$, if there exist $x_i \in \mathbb{R}^{n_i}$, $i \in [1, r]$ such that $x = \ltimes_{i=1}^r x_i$, then x is called a hypervector of order r and dimension $n_1 \times \cdots \times n_r$. Here, x_i, $i \in [1, r]$ are components of x.
(ii) The set of hypervectors of order r and dimension $n_1 \times \cdots \times n_r$ is denoted by $\mathbb{R}^{n_1\ltimes\cdots\ltimes n_r}$.
(iii) Assume $0 \neq x \in \mathbb{R}^{n_1\ltimes\cdots\ltimes n_r}$. Let $e = \min_s\{x_s \neq 0\}$ be the head index and x_e is called the head value of x. x is called a monic hypervector, if $x_e = 1$.

Lemma 2.6 ([10]). *Assume $0 \neq x \in \mathbb{R}^{n_1\ltimes\cdots\ltimes n_r}$ with the head index e,*

$$e = \min\{i \mid x_i \neq 0\}.$$

$x_e = a \neq 0$. Then there exists a unique decomposition

$$x = a \ltimes_{i=1}^r x_i, \tag{11}$$

where x_i, $i \in [1, r]$ are all monic.

In [10], a monic decomposition algorithm is obtained. By using the above lemma, the following result is obvious.

Proposition 2.7. *Consider $x \in \mathbb{R}^{\overbrace{n\ltimes\cdots\ltimes n}^{r}}$. Then x can uniquely determine a subspace $S(x) = \text{Span}\{x_1, \dots, x_r\} \subset \mathbb{R}^n$, where $x = a \ltimes_{i=1}^{r} x_i$ is its monic decomposition.*

2.4. *U-eigenvalue/eigenvector of hypermatrix*

As we emphasized in (iii) of Remark 2.4, when we mention the eigenvalues/eigenvectors of a hypermatrix, its matrix expression must be specified.

Definition 2.8. Assume $\mathcal{A} = (a_{i_1,\dots,i_d}) \in \mathbb{R}^{\overbrace{n\times\cdots\times n}^{d}}$ is given with its matrix expression

$$M_A := M^{\mathbf{j}\times\mathbf{k}}(\mathcal{A}),$$

where $\mathbf{i} = \mathbf{j} \cup \mathbf{k}$ is a partition with $|\mathbf{j}| := r \geq 1$ and $|\mathbf{k}| := s \geq 1$. Then $0 \neq x \in \mathbb{R}^{\overbrace{n\ltimes\cdots\ltimes n}^{r}}$ is called a (U-) eigenvector of $\mathcal{A}$ w.r.t. (U-) eigenvalue λ and type $B = (B_1, \dots, B_s)$, if there exist

$$y_j = B_j x, \quad j \in [1, s], \tag{12}$$

such that

$$M_A x = \lambda \ltimes_{j=1}^{s} y_j. \tag{13}$$

Remark 2.9. (i) The geometric meaning of U-eigenvalue/eigenvector is: $\mathcal{A}$ maps a subspace $S(x)$ to its (B-) related subspace $S(y)$.

(ii) The U-eigenvalue/eigenvector of $\mathcal{A}$ depends on the index partition $\mathbf{i} = \mathbf{j} \times \mathbf{k}$. Particularly, if $|\mathbf{j}| = r$, $|\mathbf{k}| = s$, and $|\mathbf{i}| = r + s$. Then $\dim(S(x)) \leq r$, $\dim(S(y)) \leq s$.

We give a simple numerical example.

Example 2.10. Consider a cubic matrix $\mathcal{A} = (a_{s,i,j}) \in \mathbb{R}^{2\times2\times2}$. It is commonly expressed in a normal form as [13]

$$M_A := M^{si\times j}(\mathcal{A}) := \begin{bmatrix} A^{(1)} \\ A^{(2)} \end{bmatrix},$$

where

$$A^{(1)} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} 0 & -1 \\ 0 & -2 \end{bmatrix}.$$

(i) Let $y_1 = y_2 = x$. Then we have

$$M_A x = \lambda x^2. \tag{14}$$

Note that $y_1 = y_2 = x$ is equivalent to

$$y_i = Bx^2 = \Xi^e_{1,2} x^2, \quad i = 1, 2. \tag{15}$$

(ii) Let

$$N_A = M^{i\times sj}(\mathcal{A}) = [A^{(1)}, A^{(2)}].$$

We consider $y = (x_1 + x_2)/2$. Then we have

$$N_A x_1 x_2 = \lambda(x_1 + x_2)/2. \tag{16}$$

Particularly, set $x_1 = x_2$, then we have

$$N_A x^2 = \lambda x, \tag{17}$$

which is the E- (or Z-) eigen-problem.

(iii) Set $x_1 = x_2 := x = (x^1, x^2)^{\mathrm{T}}$. Then

$$x^2 = ((x^1)^2, x^1x^2, x^2x^1, (x^2)^2)^{\mathrm{T}}.$$

Choose

$$y^1 = (1, 0, 0, 0)x^2 = (x^1)^2,$$
$$y^2 = (0, 0, 0, 1)x^2 = (x^2)^2.$$

Then the problem becomes H-eigen problem.

It is not difficult to choose proper B such that the U-eigen-problem becomes the eigen-problems of (5)–(7) respectively.

(iv) Using the method proposed in [10] to solve the eigen-problem. We skip the detail and give some easily verifiable results.

(a) A solution of the eigen-problem described by (14) is

$$\lambda = 2, \quad \text{and} \quad x = (1, 0)^{\mathrm{T}}. \tag{18}$$

(b) A solution of the eigen-problem described by (17) is

$$\lambda = 2, \quad \text{and} \quad x^2 = (1, 0, 0, 0)^{\mathrm{T}}. \tag{19}$$

3. Kronecker Product of Hypermatrices

Definition 3.1. Given $\mathcal{A} = (a_{i_1,\dots,i_r}) \in \mathbb{R}^{\overbrace{m\times\cdots\times m}^{r}}$ and $\mathcal{B} = (b_{j_1,\dots,j_s}) \in \mathbb{R}^{\overbrace{n\times\cdots\times n}^{s}}$. Define

$$\mathcal{C} = \{c_{i_1,\dots,i_r,j_1,\dots,j_s}\} = \mathcal{A} \otimes \mathcal{B} \in \mathbb{R}^{\overbrace{m\times\cdots\times m}^{r}\times\overbrace{n\times\cdots\times n}^{s}},$$

as follows:

$$c_{i_1,\dots,i_r,j_1,\dots,j_s} = a_{i_1,\dots,i_r} b_{j_1,\dots,j_s}, \quad \forall\, \mathbf{i}, \mathbf{j}. \tag{20}$$

The following proposition comes from Definition 3.1.

Proposition 3.2. *Let $\mathbf{i} = \mathbf{i}^1 \cup \mathbf{i}^2$, $\mathbf{j} = \mathbf{j}^1 \cup \mathbf{j}^2$ be two partitions. Assume $\mathcal{C} = \mathcal{A} \otimes \mathcal{B}$, and*

$$A = M^{\mathbf{i}^1\times\mathbf{i}^2}(\mathcal{A}),$$
$$B = M^{\mathbf{j}^1\times\mathbf{j}^2}(\mathcal{B}),$$
$$C = M^{\{\mathbf{i}^1,\mathbf{j}^1\}\times\{\mathbf{i}^2,\mathbf{j}^2\}}(\mathcal{C}),$$

then

$$C = A \otimes B. \tag{21}$$

Proposition 3.2 leads to the following result immediately.

Proposition 3.3. *Let $\mathcal{A}, \mathcal{B}$, and $\mathcal{C} = \mathcal{A} \otimes \mathcal{B}$ be as described above. Assume λ and μ are eigenvalues of $\mathcal{A}$ and $\mathcal{B}$ w.r.t. partitions $\mathbf{i} = \mathbf{i}^1 \cup \mathbf{i}^2$, $\mathbf{j} = \mathbf{j}^1 \cup \mathbf{j}^2$ as described in Proposition 3.2, and the corresponding eigenvectors are x and y respectively w.r.t. types B^1 and B^2 respectively.*

Then $\lambda\mu$ is an eigenvalue of C w.r.t. the partition

$$\{\mathbf{i}, \mathbf{j}\} = \{\mathbf{i}^1, \mathbf{j}^1\} \cup \{\mathbf{i}^2, \mathbf{j}^2\},$$

the eigenvector xy and the type $B^1 \otimes B^2$.

Example 3.4. Recall Example 2.10. Assume

$$C = M_A \otimes N_A = \begin{bmatrix} 4 & 2 & 0 & -2 & 2 & 1 & 0 & -1 \\ 0 & 2 & 0 & -4 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & -2 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & -4 & -2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 4 \end{bmatrix}.$$

Note that $x = (1, 0)^{\mathrm{T}}$ is the eigenvector of M_A with eigenvalue $\lambda_1 = 2$, and $x^2 = (1, 0.0.0)^{\mathrm{T}}$ is the eigenvector of N_A with eigenvalue $\lambda_2 = 2$ w.r.t. $y^1 = B^1 x = x$ and $y_i^2 = B^2 x^2 = x$, $i = 1, 2$. Then

$$B^1 \otimes B_1^2 \otimes B_2^2 x^3 = x^3.$$

According to Proposition 3.3 we have that $x^3 = xx^2$ is the eigenvector of C w.r.t. $\lambda = \lambda_1 \lambda_2 = 4$.

A straightforward computation shows that

$$Cx^3 = 4x^3,$$

which verified Proposition 3.3.

4. Similarity

Definition 4.1. Let $\mathcal{A} = (a_{i_1,\dots,i_d})$, $\mathcal{B} = (b_{i_1,\dots,i_d}) \in \mathbb{R}^{\overbrace{n \times \cdots \times n}^{d}}$, $\mathbf{i} = \mathbf{j} \cup \mathbf{k}$ be a partition with $|\mathbf{j}| = p$, and $|\mathbf{k}| = q$, $p + q = d$.

$\mathcal{A}$ is similar to $\mathcal{B}$ w.r.t. $\mathbf{j} \times \mathbf{k}$, denoted by $\mathcal{A} \sim_{\mathbf{j} \times \mathbf{k}} \mathcal{B}$, if there exists a nonsingular matrix $T \in \mathcal{M}_{n \times n}$, such that

$$(\underbrace{T^{-1} \otimes \cdots \otimes T^{-1}}_{s}) A (\underbrace{T \otimes \cdots \otimes T}_{r}) = B, \tag{22}$$

where

$$A = M^{\mathbf{j} \times \mathbf{k}}(\mathcal{A}), \quad B = M^{\mathbf{j} \times \mathbf{k}}(\mathcal{B}).$$

Remark 4.2. (i) Let $E = (e_1, \dots, e_n)$ be a basis of $\mathbb{R}^n$, and x be a vector expression of $X = \sum_{i=1}^n x_i e_i$. That is, $X = Ex$. Similarly, $E' = (e'_1, \dots, e'_n)$ is another basis of $\mathbb{R}^n$, and $E = E'T$. Then the expression

$$Ex \xrightarrow{A} EAx, \tag{23}$$

can be changed to

$$E'Tx \xrightarrow{A} E'TAx.$$

Setting $z = Tx$ yields

$$E'z \xrightarrow{A} E'TAT^{-1}z. \tag{24}$$

Equivalently, we have

$$E'z \xrightarrow{A} E'A'z, \tag{25}$$

where

$$A' = TAT^{-1}. \tag{26}$$

(26) shows that under the basis transformation $E = E'T$, the matrix expression of linear mapping changed from A to its similar matrix TAT^{-1}. Hence the physical meaning of “similarity” can be understood as the different expressions caused by basis transformation.

(ii) Consider a tensor $t \in \mathcal{T}_s^r$ which is a tensor over $V = \mathbb{R}^n$ with basis E and dual basis $D = E^{-1}$. $X_i \in V$, $i \in [1, r]$ are expressed by $X_i = Ex_i$ and $\Omega_j \in V^*$, $j \in [1, s]$ are expressed by $\Omega_j = \omega_j D$. Then

$$t(x_1, \dots, x_r, \omega_1, \dots, \omega_s) = \omega_s \cdots \omega_1 M_t x_1 \cdots x_r,$$

where M_t is the structure matrix of t, which is constructed as follows. Let

$$E = [e_1, e_2, \dots, e_n],$$

$$D = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}.$$

Define tensor coefficients as

$$\mu_{j_1,\dots,j_s}^{i_1,\dots,i_r} := t(e_{i_1}, \dots, e_{i_r}, d_{j_1}, \dots, d_{j_s}),$$
$$i_\alpha, j_\beta \in [1, n], \quad \alpha \in [1, r], \quad \beta \in [1, s].$$

Then we can construct the structure matrix of t as

$$M_t := \begin{bmatrix} \mu_{1,\dots,1}^{1,\dots,1} & \mu_{1,\dots,1}^{1,\dots,2} & \cdots & \mu_{1,\dots,1}^{n,\dots,n} \\ \mu_{1,\dots,2}^{1,\dots,1} & \mu_{1,\dots,2}^{1,\dots,2} & \cdots & \mu_{1,\dots,2}^{n,\dots,n} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n,\dots,n}^{1,\dots,1} & \mu_{n,\dots,n}^{1,\dots,2} & \cdots & \mu_{n,\dots,n}^{n,\dots,n} \end{bmatrix}. \tag{27}$$

Now assume the basis of the vector space V changes from E to E' and its dual basis changes from D to D', where

$$E = E'T; \quad D = T^{-1}D'.$$

Then

$$X = Ex = E'Tx := E'z \Rightarrow z = Tx;$$
$$\Omega = \omega D = \omega T^{-1}D' := \mu D' \Rightarrow \mu = \omega T^{-1}. \tag{28}$$

Assume under new basis the structure matrix becomes M'_t. Then we have

$$\mu_s \cdots \mu_1 M'_t z_1 \cdots z_r = \omega_s \cdots \omega_1 M_t x_1 \cdots x_r. \tag{29}$$

Using (28) we have

$$\omega_s \cdots \omega_1 (\underbrace{T^{-1} \otimes \cdots \otimes T^{-1}}_{s}) M'_t (\underbrace{T \otimes \cdots \otimes T}_{r}) x_1 \cdots x_r$$
$$= \omega_s \cdots \omega_1 M_t x_1 \cdots x_r. \tag{30}$$

Since ω_j and x_i are arbitrary, (30) implies that

$$(\underbrace{T^{-1} \otimes \cdots \otimes T^{-1}}_{s}) M'_t (\underbrace{T \otimes \cdots \otimes T}_{r}) = M_t. \tag{31}$$

(31) shows the physical meaning of the similarity defined by Definition 4.1. When a hypermatrix is considered as the structure matrix of a tensor, similarity means the transfer of structure matrix under basis change of the bearing space. This is obviously the generalization of the similarity of matrices.

Proposition 4.3. *Assume $\mathcal{P} = (p_{i_1,\dots,i_d})$, $\mathcal{Q} = (q_{i_1,\dots,i_d}) \in \mathbb{R}^{\overbrace{n\times\cdots\times n}^{d}}$, $\mathbf{i} = \mathbf{j} \cup \mathbf{k}$, $|\mathbf{j}| = s$, $|\mathbf{k}| = r$, and*

$$\mathcal{P} \sim_{\mathbf{j}\times\mathbf{k}} \mathcal{Q}.$$

Let $x = \ltimes_{i=1}^r x_i$ be a hyper eigenvector of $\mathcal{P}$ w.r.t. eigenvalue λ and type $B = (B_1, \dots, B_s)$. Then $z = \ltimes_{i=1}^r (T^{-1}x_i)$ is a hyper eigenvector of $\mathcal{Q}$ w.r.t. eigenvalue λ and type $B' = (B'_1, \dots, B'_s)$ with

$$B'_j = T^{-1} B_j (\underbrace{T \otimes \cdots \otimes T}_{s}), \quad j \in [1, s], \tag{32}$$

where $T \in \mathcal{M}_{n\times n}$ is nonsingular.

Proof. Let

$$P = M^{\mathbf{j}\times\mathbf{k}}(\mathcal{P}),$$
$$Q = M^{\mathbf{j}\times\mathbf{k}}(\mathcal{Q}).$$

By definition of similarity, there is a nonsingular matrix $T \in \mathcal{M}_{n\times n}$ such that

$$Q = (\underbrace{T^{-1} \otimes \cdots \otimes T^{-1}}_{s}) P (\underbrace{T \otimes \cdots \otimes T}_{t}). \tag{33}$$

Let $x = \ltimes_{i=1}^r x_i$ be the eigenvector of $\mathcal{P}$ w.r.t. eigenvalue μ and type $B = (B_1, \dots, B_s)$. That is,

$$\begin{cases} P \ltimes_{i=1}^r x_i = \lambda \ltimes_{j=1}^s y_j, \\ y_j = B_j x = B_j \ltimes_{i=1}^r x_i, \quad j \in [1, s]. \end{cases} \tag{34}$$

Consider

$$Qz = Q \ltimes_{i=1}^r (T^{-1}x_i) = (\underbrace{T^{-1} \otimes \cdots \otimes T^{-1}}_{r}) P \ltimes_{i=1}^r x_i$$
$$= \lambda (\underbrace{T^{-1} \otimes \cdots \otimes T^{-1}}_{s}) \ltimes_{j=1}^s y_j$$
$$= \lambda \ltimes_{j=1}^s (T^{-1}y_j) := \lambda \ltimes_{j=1}^s \bar{y}_j.$$
$$\bar{y}_j = T^{-1}y_j = T^{-1}B_j x$$
$$= T^{-1}B_j (\underbrace{T \otimes \cdots \otimes T}_{r}) z, \quad j \in [1, s],$$

which proves (32). □

Example 4.4. Recall Example 2.10.

(i) Let $\mathcal{B} \sim_{\mathbf{j}\times\mathbf{k}} \mathcal{A}$, where $\mathbf{j} = \{s, i\}$ and $\mathbf{k} = \{j\}$, be defined by

$$M_B = (T^{-1} \otimes T^{-1}) M_A T, \tag{35}$$

where

$$T = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then

$$M_B = \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -2 & 0 \\ -3 & -5 \end{bmatrix}.$$

Consider (14). It is known that $x = (1, 0)^T$ is the eigenvector of $\mathcal{A}$ with respect to eigenvalue 2. Set

$$z = T^{-1}x = (1, -1)^T.$$

We known that (refer to (15))

$$y_i = \Xi^e_{1,2} x^2 = (I_2 \otimes \mathbf{1}_2^T) x^2 := Bx^2, \quad i = 1, 2.$$

Under new frame, it is easy to calculate that

$$y'_i = T^{-1}B(T \otimes T) z^2 = z.$$

Hence, we have to verify that

$$M_B z = \lambda z^2. \tag{36}$$

(36) is straightforward verifiable. This verifies Proposition 4.3.

(ii) Let $\mathcal{B} \sim_{\mathbf{j}\times\mathbf{k}} \mathcal{A}$, where $\mathbf{j} = \{s\}$ and $\mathbf{k} = \{i, j\}$, be defined by

$$N_B = T^{-1} N_A (T \otimes T). \tag{37}$$

Then

$$N_B = \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -2 & 0 \\ -3 & -5 \end{bmatrix}.$$

Consider (17). It is known that $x^2 = (1,0,0,0)^T$ is the eigenvector of $\mathcal{A}$ with respect to eigenvalue 2. Set

$$z = T^{-1}x = (1,-1)^T.$$

We known that

$$y_i = \Xi^e_{1,2} x^2 = x, \quad i = 1,2.$$

Hence, we have to verify that

$$N_B z^2 = \lambda z. \tag{38}$$

(38) is straightforward verifiable. It also verifies Proposition 4.3.

5. Generalized Cayley–Hamilton Theorem for Hypermatrix

5.1. *t-product based Cayley–Hamilton theorem*

Consider the set of cubic matrices $\mathbb{R}^{r\times n\times n}$.

Definition 5.1 ([13]). Let $\mathcal{A} \in \mathbb{R}^{r\times n\times n}$.

(i) The (standard) matrix expression of $\mathcal{A} = (a_{k,i,j}) \in \mathbb{R}^{r\times m\times n}$ is defined as

$$A = M^{\{k,i\}\times\{j\}}(\mathcal{A}) = \begin{bmatrix} A^{(1)} \\ A^{(2)} \\ \vdots \\ A^{(r)} \end{bmatrix}. \tag{39}$$

(ii) Assume A is the matrix expression of $\mathcal{A} \in \mathbb{R}^{r\times m\times n}$ as in (39). The bcirc of $\mathcal{A}$ (or briefly, A) is

$$\boxplus(A) = \begin{bmatrix} A^{(1)} & A^{(r)} & A^{(r-1)} & \cdots & A^{(2)} \\ A^{(2)} & A^{(1)} & A^{(r)} & \cdots & A^{(3)} \\ A^{(3)} & A^{(2)} & \cdots & A^{(4)} & \\ \vdots & & \cdots & & \\ A^{(r)} & A^{(r-1)} & \cdots & A^{(1)} & \end{bmatrix} \in \mathcal{M}_{rm\times rn}. \tag{40}$$

(iii) Assume A is the matrix expression of $\mathcal{A} \in \mathbb{R}^{r\times m\times n}$ and B is the matrix expression of $\mathcal{B} \in \mathbb{R}^{r\times n\times q}$, the t-product of $\mathcal{A}$ and $\mathcal{B}$ is defined by

$$\mathcal{A} \circledast \mathcal{B} := \boxplus(A)B \in \mathcal{M}_{rm\times q}. \tag{41}$$

We need some new notations as follows:

(i) Define the matrix expression of the identity as

$$I^r_n = \begin{bmatrix} I_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{42}$$

(ii) Let $\mathcal{A} \in \mathbb{R}^{s\times n\times n}$. Define

$$\mathcal{A}^{\langle n\rangle} = \begin{cases} I^r_n, & n = 0, \\ \underbrace{A \circledast \cdots \circledast A}_{n}, & n \geq 1. \end{cases} \tag{43}$$

The following result is called the t-product Based Cayley–Hamilton theorem.

Proposition 5.2. *Consider $\mathcal{A} \in \mathbb{R}^{r\times n\times n}$ with its matrix expression A. Let*

$$f(\lambda) = \lambda^{rn} + c_1\lambda^{rn-1} + \cdots + c_{rn-1}\lambda + c_{rn} = 0,$$

be the characteristic function of $\boxplus(A)$. Then

$$f(\langle A\rangle) = A^{\langle rn\rangle} + c_1 A^{\langle rn-1\rangle} + \cdots + c_{rn-1}A + c_{rn}I^r_n = 0. \tag{44}$$

The t-product based Cayley–Hamilton theorem can be used for $k > 3$ cases, if the set of hypermatrices can be converted into the required cubic form as in Proposition 5.2. The following example shows this.

Remark 5.3. (i) Consider $\mathbb{R}^{\overbrace{n\times\cdots\times n}^{d}}$ with index set $\mathbf{i} = \{i_1,\dots,i_d\}$. Splitting $\mathbf{i} = \mathbf{j} \cup \mathbf{k}$, where $|\mathbf{j}| = r$, $|\mathbf{k}| = s$, $r+s = d$, and $r \geq s$, and setting $n_0 = n^s$ and $r_0 = n^{r-s}$, then $\mathcal{A} \in \mathbb{R}^{\overbrace{n\times\cdots\times n}^{d}}$ can be embedded into $\mathbb{R}^{r_0\times n_0\times n_0}$ as

$$A = M^{\mathbf{r}\times\mathbf{s}}(\mathcal{A}) = \begin{bmatrix} A^{(1)} \\ A^{(2)} \\ \vdots \\ A^{(r_0)} \end{bmatrix}.$$

With the t-product $\mathcal{C} = \mathcal{A} \circledast \mathcal{B}$ has its matrix expression as $C = A \circledast B$, the t-product Cayley–Hamilton theorem can also be obtained for $\mathbb{R}^{\overbrace{n\times\cdots\times n}^{d}}$ w.r.t. the required index partition.

(ii) Consider $\mathbb{R}^{n_1\times\cdots\times n_d}$ with its index partition

$$\mathbf{i} = \mathbf{j} \cup \mathbf{s} \cup \mathbf{t}, \tag{45}$$

where $r = \prod_{i\in\mathbf{j}} n_i$, $u = \prod_{i\in\mathbf{s}} n_i$, and $v = \prod_{i\in\mathbf{t}} n_i$. (45) is called an isosceles partition if $u = v$.

Let (45) be an isosceles partition of $\mathbb{R}^{n_1\times\cdots\times n_d}$ and $\mathcal{A} \in \mathbb{R}^{n_1\times\cdots\times n_d}$. Then the matrix expression of $\mathcal{A}$ w.r.t.

(45) is

$$A = M^{\{\mathbf{j},\mathbf{s}\}\times\{\mathbf{t}\}}(\mathcal{A}) = \begin{bmatrix} A^{(1)} \\ A^{(2)} \\ \vdots \\ A^{(r)} \end{bmatrix}. \tag{46}$$

In this way, $\mathcal{A}$ is embedded into the set of cubic matrices $\mathbb{R}^{r\times u\times u}$. The t-product Cayley–Hamilton theorem can also be obtained for $\mathbb{R}^{n_1\times\cdots\times n_d}$ w.r.t. each isosceles partition.

(iii) A trivial isosceles partition is

$$\mathbf{i} = \mathbf{i} \cup \emptyset \cup \emptyset.$$

Corresponding to this isosceles partition, the matrix expression of $\mathcal{A}$ is

$$A = V(\mathcal{A}) := (a_1, a_2, \dots, a_{\mathrm{n}})^{\mathrm{T}},$$

where $\mathrm{n} = \prod_{i=1}^{d} n_i$. Its bcirc mapping is

$$\boxplus(A) = \begin{bmatrix} a_1 & a_{\mathrm{n}} & \cdots & a_2 \\ a_2 & a_1 & \cdots & a_3 \\ \vdots & & & \\ a_{\mathrm{n}} & a_{\mathrm{n}-1} & \cdots & a_1 \end{bmatrix}. \tag{47}$$

Hence, corresponding to this trivial isosceles partition, each $\mathcal{A} \in \mathbb{R}^{n_1\times\cdots\times n_d}$ satisfies its t-product Cayley–Hamilton equation. In other words, each vector satisfies its t-product Cayley–Hamilton equation.

5.2. *DK-STP-based Cayley–Hamilton theorem*

Consider $\mathbb{R}^{n_1\times\cdots\times n_d}$ with one of its index partitions as $\mathbf{i} = \mathbf{j}\cup\mathbf{k}$, and denote $\prod_{i\in\mathbf{j}} n_i = u$, $\prod_{i\in\mathbf{k}} n_i = v$. Let $\mathcal{A} \in \mathbb{R}^{n_1\times\cdots\times n_d}$ and its matrix expression w.r.t. the partition as

$$A = M^{\mathbf{j}\times\mathbf{k}}(\mathcal{A}). \tag{48}$$

Fixing this partition, the DK-STP is defined as follows.

Definition 5.4. Let $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{n_1\times\cdots\times n_d}$, the DK-STP of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{C} = \mathcal{A} \ltimes \mathcal{B}$, is defined by

$$C = M^{\mathbf{j}\times\mathbf{k}}(\mathcal{C}) = A \ltimes_{v\times u} B = A\Psi_{v\times u}B. \tag{49}$$

Define $\Pi_A := A\Psi_{v\times u} \in \mathcal{M}_{u\times u}$, then [14]

$$A \ltimes x = \Pi_A x, \quad \forall\, x \in \mathbb{R}^u. \tag{50}$$

Define

$$A^{(k)} = \underbrace{A \ltimes \cdots \ltimes A}_{k}, \quad k \geq 1. \tag{51}$$

Then we have the following DK-STP-based Cayley–Hamilton Theorem for any hypermatrix.

Proposition 5.5. *Consider $\mathcal{A} = (a_{i_1,\dots,i_d}) \in \mathbb{R}^{n_1\times\cdots\times n_d}$, where $\mathbf{i} = \mathbf{j}\cup\mathbf{k}$ is a fixed partition, and denote $\prod_{i\in\mathbf{j}} n_i = u$, $\prod_{i\in\mathbf{k}} n_i = v$. Moreover, the matrix expression A of $\mathcal{A}$ is as in* (48)*. Assume the characteristic function of Π_A is*

$$\lambda^u + c_1\lambda^{u-1} + \cdots + c_{u-1}\lambda + c_u.$$

Then

$$A^{(u+1)} + c_1 A^{(u)} + \cdots + c_{u-1}A^{(2)} + c_u A = 0. \tag{52}$$

In fact, it is an immediate consequence of the DK-STP-based Cayley–Hamilton Theorem for nonsquare matrices [14].

Example 5.6. Recall Example 2.10 and set $A = M_{\mathcal{A}}$. It is easy to calculate that

$$\Psi_{2\times 4} = (I_2 \otimes \mathbf{1}_2^{\mathrm{T}})I_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

It follows that

$$\Pi_A = A\Psi_{2\times 4} = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -2 & -2 \end{bmatrix}.$$

The characteristic function of Π_A is

$$\lambda^4 + \lambda^3 - 6\lambda^2 = 0.$$

It is a straightforward computation to show that

$$A^{(5)} + A^{(4)} - 6A^{(3)} = 0,$$

which verifies Proposition 5.5.

5.3. *U-eigenequation based Cayley–Hamilton theorem*

Both t-product and DK-STP based Cayley–Hamilton theorems are based on classical eigenequation of square matrix. Can the U-Eigenequation be used to construct its corresponding Cayley–Hamilton theorem? The goal of this subsection is to solve this problem.

Consider $\mathcal{A} = (a_{i_1,\dots,i_d}) \in \mathbb{R}^{n_1\times\cdots\times n_d}$ and a fixed partition $\mathbf{i} = \mathbf{j}\cup\mathbf{k}$. Assume $|\mathbf{j}| = u$ and $|\mathbf{k}| = v$, $\prod_{i\in\mathbf{j}} n_i = m$, and $\prod_{i\in\mathbf{k}} n_i = n$. We set

$$\mathcal{H}_{m\times n} := \{M^{\mathbf{j}\times\mathbf{k}}(\mathcal{A}) \mid \mathcal{A} \in \mathbb{R}^{n_1\times\cdots\times n_d}\}. \tag{53}$$

It is obvious that under conventional addition and scalar multiplication $\mathcal{H}_{m\times n} \subset \mathcal{M}_{m\times n}$ is a vector space. We need to pose a "product" on $\mathcal{H}_{m\times n}$ (or $\mathcal{M}_{m\times n}$) to make it a ring.

Definition 5.7. Given a matrix $0 \neq \Psi \in \mathcal{M}_{n\times m}$. The bridge matrix (or Ψ) depending DK-STP is defined as follows.

$$A \ltimes_\Psi B := A\Psi B, \quad A, B \in \mathcal{H}_{m\times n}. \tag{54}$$

The following properties are easily verifiable.

Proposition 5.8. *Consider* $H_{m\times n}$ *with DK-STP* $\bar{\ltimes} := \bar{\ltimes}_\Psi$. *Then*

(i) $V := (H_{m\times r}, +, \cdot)$ *is a vector space, where* $+$ *and* $\cdot$ *are conventional addition and scalar product respectively.*
(ii) $R := (H_{m\times n}, \bar{\ltimes}, +)$ *is a ring.*
(iii) $(R, V, \bar{\ltimes})$ *is a module.*

Definition 5.9. A formal identity $I_{m\times n}$ w.r.t. Ψ is defined by satisfying the following.

(i)

$$I_{n\times m} \bar{\ltimes}_\psi I_{n\times m} = I_{n\times m}. \tag{55}$$

(ii)

$$A \bar{\ltimes}_\psi I_{n\times m} = I_{n\times m} \bar{\ltimes}_\psi = A, \quad \forall\, A \in \mathcal{M}_{n\times m}. \tag{56}$$

(iii)

$$I_{n\times m} \bar{\ltimes}_\Psi x = x, \quad \forall\, x \in \mathbb{R}^n. \tag{57}$$

(Equivalently, $I_{n\times m}\Psi_{m\times n} = I_n$.)

By using $\bar{\ltimes} = \bar{\ltimes}_\Psi$ and the formal identity $I_{m\times n}$, the analytic functions of $f(x)$ on $A \in \mathcal{H}_{m\times n}$ can be defined as

$$f(A) = f(0) I_{m\times n} + \sum_{n=1}^{\infty} \frac{A^{(n)}}{n!} f^{(n)}(0), \tag{58}$$

where

$$A^{(n)} := \begin{cases} I_{m\times n}, & n = 0,\\ A, & n = 1,\\ \underbrace{A \bar{\ltimes} \cdots \bar{\ltimes} A}_{n}, & n > 1. \end{cases} \tag{59}$$

Example 5.10. Given $A \in \mathcal{H}_{m\times n}$, we consider the following dynamic system:

$$\dot{x}(t) = A \bar{\ltimes} x(t), \quad x(t) \in \mathcal{H}_{m\times r}. \tag{60}$$

Let $x_0 \in \mathcal{H}_{m\times r}$, then a straightforward verification shows that the solution of (60) is

$$x(t) = e^{(At)} \bar{\ltimes} x_0,$$

where

$$e^{(At)} = I_{m\times n} + \sum_{n=1}^{\infty} \frac{t^n}{n!} A^{(n)}.$$

Now we are ready to deduce the Cayley–Hamilton Theorem of Hypermatrix.

Let $\mathcal{A} = (a_{i_1,\ldots,i_d}) \in \mathbb{R}^{\overbrace{n\times\cdots\times n}^{d}}$ with index set $\mathbf{i} = \mathbf{j} \cup \mathbf{k}$, where $|\mathbf{i}| = d$, $|\mathbf{j}| = s$, $|\mathbf{k}| = r$, $s + r = d$. and

$$A = M^{\mathbf{j}\times\mathbf{k}}(\mathcal{A}).$$

Consider the eigenequation

$$(A - \lambda B)x = 0, \tag{61}$$

where $B \in \mathcal{M}_{n^s\times n^r}$ is the matrix expression of type $\mathcal{B}$. We discuss the problem under the different kinds of B.

• Case 1: Assume B has full row rank.

Denote

$$F := B^{+} = B^{\mathrm{T}}(BB^{\mathrm{T}})^{-1}.$$

If λ is an eigenvalue of $A - \lambda B$ (which means λ is an essential eigenvalue), then rank$(A - \lambda B) =$ rank$(AF - \lambda I_{n^s})B < n^s$. It follows that rank$(AF - \lambda I_{n^s}) < n^s$ (because it is square and B is of full row rank). Then λ is an eigenvalue of AF. Conversely, it is not true.

Consider det$(AF - \lambda I)$. Then we have the characteristic function as

$$f(\lambda) = \lambda^\kappa + c_1\lambda^{\kappa-1} + \cdots + c_{\kappa-1}\lambda + c_\kappa, \tag{62}$$

where $\kappa = n^s$,

According to the (classical) Cayley–Hamilton Theorem we have

$$(AF)^\kappa + c_1(AF)^{\kappa-1} + \cdots + c_\kappa I_\kappa = 0. \tag{63}$$

Setting

$$\Psi = F,$$

yields:

$$A^{(\kappa+1)} + c_1 A^{(\kappa)} + \cdots + c_\kappa A = 0. \tag{64}$$

• Case 2: Assume rank$(B) = \kappa < n^s$.

Then we have row elimination matrix $E \in \mathcal{M}_{n^s\times n^s}$, such that

$$EB = \begin{bmatrix} 0 \\ B_t \end{bmatrix},$$

where $B_t \in \mathcal{M}_{\kappa\times n^s}$ has full row rank. Now consider the matrix of last κ rows of $EA - \lambda EB$, denoted by $A_t - \lambda B_t$. Define

$$A_0 = A_t E^{-1}, \quad \text{and} \quad B_0 = B_t E^{-1}.$$

Then the eigenequation becomes

$$(A_0 - \lambda B_0)x = 0.$$

Using the result obtained from Case 1, we can find $F_0 = B_0^{\mathrm{T}}(B_0 B_0^{\mathrm{T}})^{-1}$ and set

$$\Psi_0 = F_0.$$

Let

$$\det(A_0 F_0 - \lambda I_\kappa) = 0,$$

then we have the corresponding eigenequtien as (62),but now $\kappa < n^s$. Set

$$H = \begin{bmatrix} 0_{(n^s-\kappa)\times(n^s-\kappa)} & 0 \\ 0 & I_\kappa \end{bmatrix}.$$

Then it is easy to see that A_0 satisfies (64) w.r.t. $\Psi_0 = F_0 E_{2,2}$, where

$$E = \begin{bmatrix} E_{1,1} & E_{1,2} \\ E_{2,1} & E_{2,2} \end{bmatrix}.$$

Note that

$$\begin{aligned} & HE(AF - \lambda I_{n^s})E^{\mathrm{T}}H \\ &= H\left(\begin{bmatrix} A_1 \\ A_0 \end{bmatrix} F - \lambda I_\kappa\right) E^{\mathrm{T}}H \\ &= \left(\begin{bmatrix} 0 \\ A_0 \end{bmatrix} [0, F_0] - \lambda \begin{bmatrix} 0 & 0 \\ 0 & I_\kappa \end{bmatrix}\right) E^{\mathrm{T}}H \\ &= \left(\begin{bmatrix} 0 & 0 \\ 0 & A_0F_0 \end{bmatrix} - \lambda \begin{bmatrix} 0 & 0 \\ 0 & I_\kappa \end{bmatrix}\right) E^{\mathrm{T}}H \\ &= \left(\begin{bmatrix} 0 & 0 \\ 0 & A_0F_0 - \lambda I_\kappa \end{bmatrix}\right) E_{2,2}. \end{aligned}$$

It follows immediately that if A_0 satisfies (64) w.r.t $\Psi_0 = F_0 E_{2,2}$, then A satisfies (64) w.r.t.

$$\Psi = \begin{bmatrix} 0 & F_0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & E_{2,2} \end{bmatrix}. \tag{65}$$

Summarizing the above argument yields the Cayley–Hamilton Theorem of hypermatrix.

Theorem 5.11 (Cayley–Hamilton Theorem). *Consider* $\mathcal{A} = (a_{i_1,\ldots,i_d}) \in \mathbb{R}^{\overbrace{n\times\cdots\times n}^{d}}$ *with a partition* $\mathbf{i} = \mathbf{j} \cup \mathbf{k}$, *where* $|\mathbf{i}| = d$, $|\mathbf{j}| = s$, $|\mathbf{k}| = r$, *and* $s + r = d$. *Let*

$$A = M^{\mathbf{j}\times\mathbf{k}}(\mathcal{A}).$$

Then A *satisfies Cayley–Hamilton equation* (64), *w.r.t.*

$$\Psi = \begin{cases} F, & B \textit{ is of full row rank}, \\ \Psi \textit{ in } (65), & \textit{Otherwise}. \end{cases}$$

Example 5.12. Recall Example 2.10 and set $A = N_A$. It is easy to get that

$$B = \Psi_{2\times 4} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Since B is of full row rank, we set

$$F = B^{\mathrm{T}}(BB^{\mathrm{T}})^{-1} = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0 \\ 0 & 0.5 \\ 0 & 0.5 \end{bmatrix}.$$

The characteristic function of AF is

$$\lambda^2 - 0.5\lambda - 1.25.$$

Set

$$\Psi := F,$$

then it is easy to show that

$$A^{(3)} - 0.5A^{(2)} - 1.25A = 0,$$

which verifies Theorem 5.11.

6. Conclusion

As discussed in [10], the U-eigenvalue/eigenvector of hypermatrices is the most general eigenvalue/eigenvector. It has significant geometric meanings. This paper is a follow-up work. It proved that the U-eigenvalue/eigenvector of hypermatrices has some major properties of the eigenvalue/eigenvector of matrices, which are shown as follows: (1) If a hypermatrix is the Kronecker product of two factor hypermatrices, then the eigenvalues/eigenvectors of the hypermatrix are the products of the eigenvalues/eigenvectors of its factor hypermatrices. (2) If two hypermatrices are similar, then they have the same eigenvalues/eigenvectors. (3) Given an eigenequation of hypermatrix, the corresponding Cayley–Hamilton theorem remains true. All these factors prove the validity of the definition of U-eigenvalue/eigenvector for hypermatrices.

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