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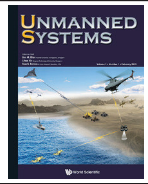

Fully Distributed Prescribed-Time Optimization of Multi-Agent Systems

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This paper investigates fully distributed prescribed-time control methods for the convex optimization of continuous-time first-order multi-agent systems (MAS). We propose a novel distributed control scheme that eliminates the necessity for global communication topology information while ensuring prescribed-time convergence to the global minimizer of the objective function. First, we design a new prescribed-time estimator to compute the average gradients and Hessians of the objective function. Concurrently, we introduce a fully distributed consensus control scheme that enables all agents to achieve consensus. Second, utilizing the estimated gradients and Hessians, and with consensus achieved among the MAS, we develop a prescribed-time optimization controller that ensures the convergence of all agents' states to the global minimizer. A numerical simulation is included to demonstrate the effectiveness of the proposed control methods.

Keywords: Fully distributed algorithms; convex optimization; prescribed-time convergence; multi-agent systems.

1. Introduction

Over the past decade, distributed optimization for MAS has garnered significant attention due to its extensive applications, including distributed sensor networks [1], distributed resource allocation [2] and large-scale machine learning [3]. In the context of distributed optimization, it implies that a global optimal solution can be attained while achieving consensus among a group of agents, under the constraint that each agent has access only to a portion of the overall objective function.

To date, there exists a considerable number of distributed optimization algorithms addressing various concerns and perspectives, including discrete-time algorithms [4–6] and continuous-time algorithms. Continuous-time approaches are applicable in coordinating the motion of MAS. For instance, multiple physical vehicles, governed by continuous-time dynamics, may need to converge at an optimal rendezvous point for the group. Consequently, numerous continuous-time algorithms have been developed. In [7, 8], a set of sufficient and necessary conditions for achieving distributed unconstrained convex optimization was presented. In [9], a second-order distributed optimization algorithm with an enhanced convergence rate was developed for weight-balanced communication networks. Numerous other issues have been studied concerning the distributed optimization of MAS, such as event-triggered mechanisms [10, 11], time-varying objective functions [12, 13], distributed multi-objective optimization [14] and communication time delays [15].

All the aforementioned control methods achieve optimal solutions through asymptotic or exponential convergence, indicating that convergence occurs over an infinite time horizon. However, in certain practical scenarios, such as smart grid management and resource allocation, convergence time is a critical performance metric for assessing an algorithm's effectiveness. Therefore, designing algorithms capable of achieving optimal solutions within a finite time frame is of considerable importance. Consequently, numerous remarkable finite-time [16–20] and fixed-time algorithms [21–24] have been presented. While the aforementioned studies achieve finite-time or fixed-time convergence, the convergence time is influenced by initial conditions and system parameters, rendering advance specification

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impossible. This constraint limits the practical applicability of these algorithms. To address this limitation, developing a prescribed-time distributed optimization approach is both theoretically and practically valuable. In [25], the authors proposed an observer-based three-stage control scheme, consisting of a group gradients and Hessians estimation stage, a consensus stage and an optimization stage, which guarantees the achievement of the minimizer of the global cost function, irrespective of the initial states of the MAS. Both two-stage continuous-time and event-based optimization algorithms were constructed in [26], independent of the gradient information of the entire group. To address distributed optimization with constraints, the authors in [27] presented a control scheme independent of the MAS's initial states and system parameters. For time-varying distributed optimization problems, [28] designed a nested structure to address the time-varying optimization issues for both single-integrator and double-integrator multi-agent systems. However, the control parameters of the aforementioned results all depend on information regarding the global communication topology, specifically the smallest nonzero eigenvalue of the Laplacian matrix, rendering the controllers not fully distributed. Fully distributed algorithms refer to the design of controller-related parameters without the need to know the communication topology parameters of the whole team. This limitation hinders the practical application of these methods. Without reliable knowledge of the communication topology, designing the control protocol encounters uncertainty in control parameters to a certain extent.

Motivated by the previous discussions, we investigate the prescribed-time control problems for distributed convex optimization of single-integrator MAS. The main contributions of this paper are summarized as follows:

(1) A fully distributed control framework is presented to achieve prescribed-time convergence without the need for any global information regarding the Laplacian matrix. This feature distinguishes the proposed approach in this paper from those presented in [25–28], where the selection of the controller gain depends on the second smallest eigenvalue of the Laplacian matrix, denoted as λ_2.
(2) The proposed two-stage control scheme ensures that each agent in the multi-agent system achieves the minimizer of the global objective function within a predetermined time frame, independent of the agents' initial conditions, which differs from many existing finite-time or fixed-time distributed optimization algorithms, such as [18–22].

The remainder of this paper is organized as follows. Section 2 presents graph theory, the system model, control objectives and several useful lemmas. Section 3 presents the main results, including fully distributed estimators of group Hessians and gradients, fully prescribed-time consensus algorithms and gradient descent algorithms. Additionally, Sec. 4 presents the simulation results that illustrate the effectiveness of the proposed control scheme. Section 5 presents the conclusions of this paper, along with prospects for future research.

Notation:
Let $\mathbb{R}_+$ and $\mathbb{R}$ denote the sets of positive real numbers and real numbers, respectively. The collection of real matrices of dimensions $n \times m$ is represented as $\mathbb{R}^{n\times m}$. $\|\cdot\|$ represents the Euclidean norm and $|\cdot|$ denotes absolute value. $1_n \in \mathbb{R}^n$ is a column vector with all elements equal to 1. I_n is a n-dimensional identity matrix. The vector x's p-norm is represented as $\|x\|_p$, where p is in the range $p \in [1, \infty]$. For a matrix $X \in \mathbb{R}^{n\times n}$, denote $\text{vec}(X) = [\chi_1^T, \dots, \chi_n^T]^T$, where χ_i is the ith column of X. $X \otimes Y$ represents the Kronecker product of matrix X and Y.

2. Problem Formulation and Background

2.1. *Problem formulation*

Consider a multi-agent system composed of n agents that communicate through a graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where the set of vertices and the edge set are represented by $\mathcal{V} = \{v_1, \dots, v_n\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, respectively. The neighborhood of agent v_i is denoted as $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_i, v_j) \in \mathcal{E}\}$, which includes all nodes that communicate with v_i. The Laplacian matrix is defined as $L = (l_{ij}) \in \mathbb{R}^{n\times n}$ with entries $l_{ij} = -a_{ij}$ for $i \neq j$ and $l_{ii} = \sum_{j=1, j\neq i}^{n} a_{ij}$, where $a_{ij} = a_{ji} = 1$ if $(v_i, v_j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. For an undirected graph, L is symmetric positive semidefinite, with eigenvalues that can be ordered as $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. The second-smallest eigenvalue λ_2 is referred to as the algebraic connectivity of graph $\mathcal{G}$. The incidence matrix of the graph is defined as $B = [b_{ij}] \in \{-1, 0, 1\}^{n\times \ell}$, where $b_{ij} = 1$ if edge e_j enters node v_i, $b_{ij} = -1$ if edge e_j leaves node v_i and $b_{ij} = 0$ otherwise; here, ℓ represents the number of edges. For the undirected graph $\mathcal{G}$, the direction of an edge can be set arbitrarily. In the column corresponding to an edge in the incidence matrix B, one row contains a 1 corresponding to one vertex, another row contains a -1 corresponding to the other vertex, while all remaining rows contain 0.

Assumption 1. The communication graph $\mathcal{G}$ is connected and undirected.

Consider a multi-agent system comprising n agents:

$$\dot{x}_i(t) = u_i(t), \quad i \in \mathcal{I}. \tag{1}$$

Define the state vector as $x(t) = [x_1, \dots, x_n]^T \in \mathbb{R}^{nm}$ and the control input vector as $u(t) = [u_1, \dots, u_n]^T \in \mathbb{R}^{nm}$, where $u_i(t)$ denotes the control input for agent i, and m represents the dimension of each agent's state x_i.

Let $f_i : \mathbb{R}^m \to \mathbb{R}$ denote the objective function of the ith agent. The optimization problem and consensus constraint can be expressed as follows:

$$\begin{cases} \min_{x_1,x_2,\ldots,x_n} \sum_{i=1}^{n} f_i(x_i), & i \in \mathcal{I}, \\ \text{subject to} \quad x_i = x_j, & \forall\, i,j \in \mathcal{I}. \end{cases} \tag{2}$$

This ensures that all agents agree on the consensus state $x^\star$, which also satisfies the following assumption.

Assumption 2. The sum of local objective functions $\sum_{i=1}^{n} f_i(x_i)$ is strongly convex. The distributed optimization problem (2) has a unique solution, denoted as $x^\star$, which serves as its minimizer.

In contrast to previous works such as [29, 30], which require all agents' initial states to meet specific conditions, our control design relaxes this requirement. This design ensures that the control protocol operates irrespective of the agents' initial states. Define $\nabla F(x) = \sum_{i=1}^{n} \nabla f_i(x_i) \in \mathbb{R}^m$ and $\nabla^2 F(x) = \sum_{i=1}^{n} \nabla^2 f_i(x_i) \in \mathbb{R}^{m\times m}$ as the group gradient and group Hessian, respectively, with $\nabla f_j(x_j)$, and $\nabla^2 f_j(x_j)$ satisfying Assumption 3.

Now, define $\zeta_i(t) = [\nabla f_i(x_i), \mathrm{vec}(\nabla^2 f_i(x_i))]^T \in \mathbb{R}^{m+m^2}$ as the augmented vector comprising the gradient and Hessian for each agent. This vector encompasses both the first and second-order derivatives, where $\mathrm{vec}(\nabla^2 f_i(x_i))$ denotes the vectorized Hessian.

The average augmented vector is defined as

$$\bar{\zeta}(x) = \frac{1}{n}\sum_{i=1}^{n} \delta_i(t) \in \mathbb{R}^{m+m^2}. \tag{3}$$

This represents the averaged group variable comprising the gradients and Hessians across all agents.

By ensuring the convergence of the control protocol within the prescribed time, we present the following assumptions for achieving consensus at $x^\star$ from arbitrary initial states.

Assumption 3. Each agent $i \in \mathcal{I}$ has access to information from all its neighboring agents, including x_j, $\nabla f_j(x_j)$ and $\nabla^2 f_j(x_j)$ for all $j \in \mathcal{N}_i$.

Assumption 4. Each function f_i, for $i \in \mathcal{I}$, is convex and at least twice continuously differentiable. The group Hessian $\nabla^2 F(x)$ is positive definite for all $x \in \mathbb{R}^{nm}$. Both $\frac{\partial}{\partial t}\nabla F(x)$ and $\frac{\partial}{\partial t}\nabla^2 F(x)$ are bounded.

Assumption 4 is satisfied by a wide range of functions, including quadratic, fractional, trigonometric and exponential functions with negative powers, as well as other twice continuously differentiable functions. Practical applications of these functions can be found in various fields. For instance, quadratic cost functions are utilized in the economic dispatch of smart grids [31], while squared aggregate distance functions are employed in the optimal rendezvous of multiple mobile robots [32].

2.2. *Preliminaries*

The useful lemmas and definition are given as follows.

Lemma 1 (33). *The algebraic connectivity $\lambda_2 > 0$ if and only if $\mathcal{G}$ is a connected graph.*

Lemma 2 (34). *Let $\mathcal{M} = I_n - \frac{1}{n}1_n 1_n^T$. For an undirected connected graph $\mathcal{G}$, the following equation holds:*

$$\mathcal{M} = BB^T(BB^T)^+ = B(B^TB)^+B^T. \tag{4}$$

Lemma 3 (35). *Consider the following system:*

$$\dot{x}(t) = f(x(t),t), \quad x(t_0) = x_0, \tag{5}$$

where $f : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ is a nonlinear vector function. Define a scaling function as

$$\mu(t,t_0;T) = \begin{cases} \rho(t,t_0;T) \triangleq \rho, & t \in [t_0,T), \\ 0, & t \in [T,\infty), \end{cases} \tag{6}$$

satisfying the following conditions: (1) ρ and its primitive function $\bar{\rho}$ are monotonically increasing; (2) $\bar{\rho}(t_0) \in \mathbb{R}_+ < +\infty$ and $\rho(t_0) \in \mathbb{R}_+ < +\infty$; (3) $\bar{\rho} \to +\infty$ and $\rho \to +\infty$ as $t \to T$; (4) $\lim_{t\to T} \rho^2 \exp(-\lambda\bar{\rho}(t)) \in L_\infty$, where $\lambda \in \mathbb{R}_+$ is a constant. For system (5), if there exists a continuously differentiable nonnegative function $V(x,t)$ with $V(x,t) > 0$ for $x \neq 0$, $V(0,t) = 0$, and its derivative satisfies $\dot{V} \le (-\lambda_1\mu + \lambda_2)V$ with $\lambda_1 \in \mathbb{R}_+$ and $\lambda_2 \in \mathbb{R}$, then the origin of system (5) is globally prescribed-time stable with the corresponding prescribed time T.

Definition 1 (Prescribed-time stability [35]). System (5) is said to be prescribed-time convergent to zero if, for all $T \in \mathbb{R}_+$ and $\forall\, x_0 \in \mathbb{R}^n$, there exists a time T such that $\|x(t,x_0)\| = 0$ holds for $t \ge T$, where T is referred to as the prescribed time.

In this work, we choose a scaling function as

$$\mu(t,t_0;T) = \begin{cases} \beta(T-t)^{-h}, & t \in [t_0,T), \\ 0, & t \in [T,\infty), \end{cases} \tag{7}$$

where $\beta, h \in \mathbb{R}_+$ and $h > 1$.

Subsequently, we introduce the following lemma, which is essential for deriving the main results.

Lemma 4 (36, 37). *Let $g(y) \ge 0$ be a continuous function defined on the interval $y \in [a,b)$ with a flaw at $y = b$. If it satisfies*

$$\lim_{y\to b^-}(b-y)g(y) = d,$$

where d is a positive constant or $+\infty$, then the improper integral $\int_a^b g(y)\,dy$ diverges.

3. Main Results

In this section, we present the main results. A two-stage prescribed-time control protocol for system (1) is designed in the following form:

$$u_i(t) = \begin{cases} -(k_1 + \mu(t, t_0; T_1)) \\ \quad (c_i + g_i)e_i, & t \in [t_0, t_1), \\ -\mu(t, T_1; T_p)[w_{i1}, \dots, \\ \quad w_{im}]^{-1} w_{i0}^T, & t \in [t_1, +\infty), \end{cases} \tag{8}$$

where $k_1, T_1, t_1, T_p \in \mathbb{R}_+$,$T_1 \le t_1 < T_p$, $e_i = \sum_{j=1}^n a_{ij}(x_i - x_j)$, $g_i = e_i^T e_i$, c_i is updated as

$$\dot{c}_i = g_i, c_i(t_0) \ge 1 \tag{9}$$

and $w_i = [w_{i0}^T, w_{i1}^T, \dots, w_{im}^T] \in \mathbb{R}^{m+m^2}$ is the estimation of group variable, which would be designed in the Sec. 3.1.

Remark 1. During the first time interval $[t_0, t_1)$, the control term $-(k_1 + \mu(t, t_0; T_1))(c_i + g_i)e_i$ is designed to achieve consensus across the entire system. Simultaneously, the distributed estimator $[w_{i0}^T, w_{i1}^T, \dots, w_{im}^T]$ for each agent estimates the sum of individual gradients and the Hessian matrix during this period, as described in (10). In the second time interval $[t_1, +\infty)$, the control term $-\mu(t, T_1; T_p)[w_{i1}, \dots, w_{im}]^{-1} w_{i0}^T$ is employed to minimize the global cost function $\sum_{i=1}^n f_i(x_i)$ after consensus is achieved.

3.1. *Distributed prescribed-time state consensus and group variables' estimation over* $[t_0, T_1)$

We first present the design of the estimator for the objective function's gradient and Hessian. Let $w_i(t) \in \mathbb{R}^{m+m^2}$ denote agent i's estimate of the average the augmented vector of the gradient and Hessian, i.e. $\bar{\delta}(x) = \frac{1}{n}\sum_{i=1}^n \delta_i(t) \in \mathbb{R}^{m+m^2}$.

$$\begin{cases} w_i(t) = \zeta_i(t) + \delta_i(t), \\ \dot{\zeta}_i(t) = -\mu(t, t_0; T_1)\zeta_i(t) - \sum_{j \in \mathcal{N}_i} k_{ij}(t) \\ \qquad \circ \mathrm{sgn}(w_i(t) - w_j(t)), \end{cases} \tag{10}$$

where $\mu(t, t_0; T_1)$ is the time scaling function, and $k_{ij}(t)$ is updated as

$$\dot{k}_{ij}(t) = |w_i(t) - w_j(t)| \tag{11}$$

with $k_{ij}(t)(t_0) \ge 1$. For simplification, let us denote $z(t) = (B^T \otimes I_{m+m^2})w(t)$. The matrix form of (10) can be expressed as

$$\begin{cases} w(t) = \zeta(t) + \delta(t), \\ \dot{\zeta}(t) = -\mu(t, t_0; T_1)\zeta(t) - (B \otimes I_{m+m^2})K(t) \\ \qquad \times \mathrm{sgn}((B^T \otimes I_{m+m^2})w(t)), \end{cases} \tag{12}$$

where $w(t) = [w_1^T(t), \dots, w_n^T(t)]^T \in \mathbb{R}^{(m+m^2)n}$, $\zeta(t) = [\zeta_1^T(t), \dots, \zeta_n^T(t)]^T \in \mathbb{R}^{(m+m^2)n}$, $K(t) \in \mathbb{R}^{m\ell \times m\ell}$ is a diagonal matrix with diagonal entries κ_ι, $\iota \in \{1, \dots, m\ell\}$. If we denote $z(t) = [x_1, \dots, x_{m\ell}]^T = (B^T \otimes I_{m+m^2})w(t)$, then it follows that

$$\dot{\kappa}_\iota(t) = |z_\iota|. \tag{13}$$

Denote $\tilde{w}_i(t) = w_i(t) - \frac{1}{n}\sum_{i=1}^n \delta_i(t)$, and it can be written in the matrix form as

$$\begin{aligned} \dot{\tilde{w}}(t) &= \dot{w}(t) - \frac{1}{n}(1_n 1_n^T \otimes I_n)\dot{\delta}(t) \\ &= \dot{\zeta}(t) + \dot{\delta}(t) - \frac{1}{n}(1_n 1_n^T \otimes I_{m+m^2})\dot{\delta}(t) \\ &= -\mu(t, t_0; T_1)\tilde{w}(t) - (B \otimes I_{m+m^2})K(t)\,\mathrm{sgn}(x(t)) \\ &\quad + (\mathcal{M} \otimes I_{m+m^2})\dot{\delta}(t), \end{aligned} \tag{14}$$

where $\tilde{w} = [\tilde{w}_1^T, \dots, \tilde{w}_n^T]^T$. We have the following results.

Theorem 1. *Suppose Assumptions* 1, 3 *and* 4 *hold, if estimator* (10) *is applied with* $T_1 > t_0$, *then the estimation errors* $w_i(t) - \bar{\delta}(x)$ *converge to 0 within the prescribed time* T_1.

Proof. The proof can be divided into two steps. The first step involves demonstrating that $\sum_{i=1}^n \tilde{w}_i$ decays to zero within a prescribed time. The second step is to show that $\tilde{w}_i$ reach consensus in a prescribed time. □

Left multiplying (14) with $1_n^T \otimes I_{m+m^2}$ yields

$$\frac{d}{dt}(1_n^T \otimes I_{m+m^2}\tilde{w}) = -\mu(t, t_0; T_1)(1_n^T \otimes I_{m+m^2}\tilde{w}). \tag{15}$$

By applying Lemma 3, we can deduce that $1_n^T \otimes I_{m+m^2}\tilde{w}$ converges to zero within a prescribed time T_1, with constructing a Lyapunov function $V_0 = \frac{1}{2}(1_n^T \otimes I_{m+m^2}\tilde{w})^T(1_n^T \otimes I_{m+m^2}\tilde{w})$. The time derivative of V_0 along (15) is given by $\dot{V}_0 = -\mu(t, t_0; T_1)\frac{1}{2}(1_n^T \otimes I_{m+m^2}\tilde{w})^T(1_n^T \otimes I_{m+m^2}\tilde{w}) = -2\mu(t, t_0; T_1)V_0$. Based on Lemma 3, we know that $\lim_{t \to T_1} V_0 = 0$ and $V_0 = 0, \forall t \ge T_1$, which means that $1_n^T \otimes I_{m+m^2}\tilde{w} = 0$.

We now prove that the consensus of $\tilde{w}_i$ can be reached in T_1, i.e. $x(t)$ converges to zero at $t = T_1$. Consider the Lyapunov function as

$$V_2 = V_1 + \frac{1}{2}\sum_{\iota=1}^{m\ell}(\kappa_\iota(t) - \bar{\kappa})^2, \tag{16}$$

where $V_1 = \frac{1}{2}\tilde{w}^T(t)(\mathcal{M} \otimes I_{m+m^2})\tilde{w}(t) = \frac{1}{2}z^T(t)((B^T B)^+ \otimes I_{m+m^2})z(t)$, $\bar{\kappa}$ is a constant to be determined. The derivative of V can be calculated as

$$\begin{aligned}
\dot{V}_2 &= \tilde{w}^T(t)(\mathcal{M} \otimes I_{m+m^2})\dot{\tilde{w}}(t) + \sum_{\iota=1}^{m\ell}(\kappa_\iota(t) - \bar{\kappa})\dot{\kappa}_\iota(t) \\
&= -\mu(t, t_0; T_1)\tilde{w}^T(t)(\mathcal{M} \otimes I_{m+m^2})\tilde{w}(t) - \tilde{w}^T(t) \\
&\quad \times (\mathcal{M} \otimes I_{m+m^2})(B \otimes I_{m+m^2})K(t) \\
&\quad \times \mathrm{sgn}((B^T \otimes I_{m+m^2})w(t)) + \tilde{w}^T(t) \\
&\quad \times (\mathcal{M} \otimes I_{m+m^2})(\mathcal{M} \otimes I_{m+m^2})\dot{\delta}(t) \\
&\quad + \sum_{\iota=1}^{m\ell}\kappa_\iota(t)\dot{\kappa}_\iota(t) - \sum_{\iota=1}^{m\ell}(\bar{\kappa})\dot{\kappa}_\iota(t) \\
&= -\mu(t, t_0; T_1)\tilde{w}^T(t)(\mathcal{M} \otimes I_{m+m^2})\tilde{w}(t) \\
&\quad + \sum_{\iota=1}^{m\ell}k_{ij}^T(t)\dot{\kappa}_\iota(t) - \tilde{w}^T(t)(B \otimes I_{m+m^2})K(t) \\
&\quad \times \mathrm{sgn}((B^T \otimes I_{m+m^2})w(t)) + \tilde{w}^T(t) \\
&\quad \times (\mathcal{M} \otimes I_{m+m^2})(\mathcal{M} \otimes I_{m+m^2})\dot{\delta}(t) \\
&\quad - \sum_{\iota=1}^{m\ell}(\bar{\kappa})|w_i(t) - w_j(t)| \\
&= -\mu(t, t_0; T_1)\tilde{w}^T(t)(\mathcal{M} \otimes I_{m+m^2})\tilde{w}(t) \\
&\quad + \tilde{w}^T(t)(B \otimes I_{m+m^2})((B^TB)^+ \otimes I_{m+m^2})(B^T \\
&\quad \otimes I_{m+m^2}\dot{\delta}(t)) - \bar{k}\|(B^T \otimes I_{m+m^2})\tilde{w}(t)\|_1 \\
&\le -\mu(t, t_0; T_1)\tilde{w}^T(t)(\mathcal{M} \otimes I_{m+m^2})\tilde{w}(t) \\
&\quad + \|(B^T \otimes I_{m+m^2})\tilde{w}(t)\|_1(\|(B^TB)^+ \\
&\quad \otimes I_{m+m^2}\|_\infty\|B^T \otimes I_{m+m^2}\zeta(t)\|_\infty - \bar{k}), \qquad (17)
\end{aligned}$$

where we use the fact that $(B^T \otimes I_{m+m^2})\tilde{w}(t) = (B^T \otimes I_{m+m^2})w(t)$. Note that $\|B^T \otimes I_{m+m^2}\dot{\delta}(t)\|_\infty$ is bounded for a connected undirected network and according to Assumption 4, $\|B^T \otimes I_{m+m^2}\dot{\delta}(t)\|_\infty$ is also upper bounded. Thus if $\bar{k}$ is selected that

$$\bar{\kappa} \ge \|(B^TB)^+ \otimes I_{m+m^2}\|_\infty\|B^T \otimes I_{m+m^2}\dot{\delta}(t)\|_\infty, \qquad (18)$$

then we have

$$\begin{aligned}
\dot{V}_2 &\le -\mu(t, t_0; T_1)\tilde{w}^T(t)(\mathcal{M} \otimes I_{m+m^2})\tilde{w}(t) \\
&= -\mu(t, t_0; T_1)z^T(t)((B^TB)^+ \otimes I_{m+m^2})z(t) \\
&= -2\mu(t, t_0; T_1)V_1 \le 0. \qquad (19)
\end{aligned}$$

Note that V_2 is upper bounded, which means that $z(t)$ and $\kappa_\iota(t) - \bar{\kappa}$ are bounded. The following proof draws inspiration from Part B of the proof in [37, Theorem 1]. By integrating both sides of Eq. (19), we obtain $\int_0^{T_1} \dot{V}\,dt \le -2\int_0^{T_1} \mu V_1\,dt$, which can be rearranged as $\int_0^{T_1} 2\mu V_1\,dt \le C$,, where $C = V(0) - V(T_1^-)$ is a positive constant. As a result, $\int_0^T 2\mu V_1\,dt$ is bounded.

To demonstrate that $z(t)$ reaches zero within the prescribed time T_1, we use a contradiction argument. Assume, for contradiction, that $\lim_{t\to T_1^-} V_1(t) = \delta \neq 0$, where δ is a positive constant. Given that $\lim_{t\to T_1^-} \mu(t) = +\infty$, the improper integral $\int_0^{T_1} 2\mu V_1\,dt$ would become unbounded as t approaches T_1. On the one hand, since $2\mu V_1$ is nonnegative, the function $F(\tau) = \int_0^\tau 2\mu V_1\,dt$ increases monotonically on $[0, T_1)$. Because $F(\tau)$ is bounded by C, the integral $\int_0^T 2\mu V_1\,dt$ must converge. However, since $\lim_{t\to T_1^-}(T_1 - t)\mu V_1 = +\infty > 0$, this implies that the integral $\int_0^T 2\mu V_1\,dt$ is actually divergent, leading to a contradiction.

This contradiction proves that the assumption is incorrect, and thus $\lim_{t\to T^-} V_1(t) = \delta = 0$. Consequently, we conclude that $\lim_{t\to T^-} z(t) = 0$. By the continuity of $z(t)$, it follows that $z(T_1) = 0$ and $V_1(T_1) = 0$.

For $t \ge T_1$, V_1 satisfies

$$\begin{aligned}
\dot{V}_1 &= \dot{V}_2 - \sum_{\iota=1}^{m\ell}(\kappa_\iota(t) - \bar{\kappa})\dot{\kappa}_\iota(t) \\
&\le -2\mu(t)V_1 - \sum_{\iota=1}^{m\ell}(\kappa_\iota(t) - \bar{\kappa})z_\iota(t) \le 0. \qquad (20)
\end{aligned}$$

Thus, we conclude that for $t \ge T_1$, $V_1(t) = 0$ and $z(t) = 0$, completing the proof, i.e. the consensus of $\tilde{w}_i$ can be reached at time $t = T_1$. Given the previous conclusion that $\sum_{i=1}^n \tilde{w}_i$ converges to zero at time $t = T_1$, we can therefore conclude that $w_i - \frac{1}{n}\sum_{i=1}^n \delta_i(t)$ converges to the origin in prescribed time T_1.

In the same time interval $[t_0, t_1)$, applying the first control term of (8), the consensus of the MAS can also be achieved. We can directly use the results in [35], as presented in the following lemma.

Lemma 5. *For system* (1), *if Assumption* 1 *holds, applying the fully distributed control scheme $u_i = -(k_1 + \mu(t, t_0; T_1))(c_i + g_i)e_i$ with $k_1, T_1 \in \mathbb{R}_+$, then the consensus can be achieved in a prescribed time, i.e. $x_i = x_j$, $\forall\, t \ge T_1$, $i, j \in \mathcal{I}$.*

Proof. The proof can be referred to [35, Theorem 1], thus it is omitted here. □

3.2. *Prescribed-time optimization of the global objective function over* $[t_1, +\infty)$

To facilitate the development of the distributed optimization scheme, in this subsection, we first propose a centralized prescribed-time protocol that leverages the group gradient and Hessian, as stated in the following theorem.

Theorem 2. *Consider the multi-agent system* (1) *with the same inputs and initial conditions, i.e.*

$$\dot{x}_i = u^\star, \quad x_i(t_1) = x_j(t_1), \quad \forall\, i,j \in \mathcal{I}, \tag{21}$$

where the optimal control protocol is defined as

$$u^\star = -(k_2 + \mu(t, t_1; T_p))\frac{\nabla F(x)}{\nabla^2 F(x)}, \tag{22}$$

with $c_0 \in \mathbb{R}_{>0}$. From (7) *and* (8), *we can deduce that $x_i(t) = x_j(t)$ for all $i, j \in \mathcal{I}$, over the interval $[t_0, \infty)$. Moreover, the trajectories of all agents will converge to $x^\star$, which is the minimizer of the group objective function, within a predefined time frame T_p.*

Proof. First, we select the following Lyapunov function candidate:

$$V_3(t) = \frac{1}{2}\nabla F^T(x)\nabla F(x). \tag{23}$$

Since $V_3(t)$ is positive semidefinite and $V_3(t) = 0 \iff \nabla F(x) = 0$, this implies that the group objective function reaches its minimum. By differentiating (23), we obtain

$$\begin{aligned}\dot{V}_3(t) &= \nabla F^T(x)(\nabla^2 F(x) u^\star)\\ &= -(k_2 + \mu(t, t_1; T_p))\nabla F^T(x)\nabla F(x)\\ &= -2(k_2 + \mu(t, t_1; T_p))V_3(t).\end{aligned} \tag{24}$$

Using Lemma 4, we deduce that $V_3(t) = 0$ for $t \geq t_1 + T_p$, meaning $\sum_{i=1}^N f_i(x) \equiv 0$ over $[t_1 + T_p, \infty)$, which implies that all agents $x_i \equiv x^\star$ over $[t_1 + T_p, \infty)$. Consequently, all trajectories in the MAS reach the group minimizer $x^\star$ within the predefined time T_p, starting from any set of bounded initial conditions. □

Now based on Theorem 1, Lemma 5 and Theorem 2, we propose the main theorem as follows.

Theorem 3. *Consider system* (1) *controlled by*

$$u_i(t) = \begin{cases} -(k_1 + \mu(t, t_0; T_1))(c_i \\ \quad + g_i)e_i, & t \in [t_0, t_1), \\ -(k_2 + \mu(t, t_1; T_p))[w_{i1}, \ldots, \\ \quad \times w_{im}]^{-1} w_{i0}^T, & t \in [t_1, +\infty), \end{cases} \tag{25}$$

where

$$\begin{cases} \dot{c}_i = g_i, & c_i(t_0) \geq 1, \\ g_i = e_i^T e_i, & e_i = \sum_{j=1}^n a_{ij}(x_i - x_j), \end{cases} \tag{26}$$

and $w_i = [w_{i0}^T, w_{i1}^T, \ldots, w_{im}^T] \in \mathbb{R}^{m+m^2}$ is updated as

$$\begin{cases} w_i(t) = \zeta_i(t) + \delta_i(t), \\ \dot{\zeta}_i(t) = -\mu(t, t_0; T_1)\zeta_i(t) - \sum_{j \in \mathcal{N}_i} k_{ij}(t) \\ \qquad \circ sgn(w_i(t) - w_j(t)), \\ \dot{k}_{ij}(t) = |w_i(t) - w_j(t)|, \quad k_{ij}(t)(t_0) \geq 1, \\ \delta_i(t) = [\nabla f_i(x_i), vec(\nabla^2 f_i(x_i))]^T, \end{cases} \tag{27}$$

with $k_1, T_1, t_1, T_p \in \mathbb{R}_+, T_1 \leq t_1 < T_p$. If Assumptions 1–4 *hold, then the trajectories of all agents will converge to the minimizer of the global objective function $\sum_{i=1}^n f_i(x_i)$, within a predetermined time interval T_p.*

Proof. According to Lemma 5 and Theorem 1, we know that when $t \geq T_1$, the MASs have identical states and $w_i = \frac{1}{n}\sum_{i=1}^n \delta_i(t)$, thus $u_i = -\mu(t, T_1; T_p)[w_{i1}, \ldots, w_{im}]^{-1} w_{i0}^T = -\mu(t, t_1; T_p)\frac{\nabla F(x)}{\nabla^2 F(x)}$. Then based on Theorem 2, we can conclude that the distributed optimization problem (2) can be solved in a prescribed-time T_p, i.e. $x_i = x_j = x^\star, \forall t \geq T_1$, $i, j \in \mathcal{I}$. □

Remark 2. In contrast to the distributed control methods presented in [25–28], the proposed controller in (25) is fully distributed and does not rely on any global information regarding the Laplacian matrix. Control parameters k_1 and k_2 are independent of the communication graph, and their values should be chosen within reasonable bounds to maintain an efficient control design. This characteristic broadens the range of applications for the algorithm, making it more adaptable to various operational environments. Furthermore, unlike the three-stage methods outlined in [25], our algorithm is notably more concise, operating in just two stages. We integrate the estimation of the group Hessians and gradients of the objective function with the consensus process within the same time period. This streamlined approach not only enhances efficiency but also reduces communication overhead among agents.

Remark 3. Compared with traditional distributed average consensus algorithms [26–28], where the controller gain selection relies on the second smallest eigenvalue (λ_2) of the Laplacian matrix, which is a global network parameter, the proposed control scheme (8) eliminates this global information requirement.In addition, unlike the results in [29, 30], where the convergence time cannot be arbitrarily chosen, the controller in (8) allows for the convergence time to be freely specified, providing more design flexibility. Nevertheless, the control algorithm designed in this paper also has some drawbacks. First, the algorithm is adaptive, which increases the computational load. Second, the algorithm is relatively sensitive to disturbances, a common issue with prescribed-time convergence algorithms. These problems will be further investigated in subsequent research.

4. Numerical Simulations

In this section, we present numerical simulations to validate the effectiveness of the proposed controllers (25)–(27). A multi-agent system comprising six agents is considered; the dynamics are described by (10) with a dimension of $m = 2$. Each agent's objective function is defined as follows:

$$f_i(x_i) = \begin{cases} 0.5(x_i - x_a)^2, & \forall\, i \in \{1, 2\}, \\ 0.6(x_i - x_b)^2 + 0.6e^{-0.5x_i}, & \forall\, i \in \{3, 4\}, \\ 0.1(x_i - x_c)^2 + \ln(2 + 0.1x_i^2), & \forall\, i \in \{5, 6\}, \end{cases} \tag{28}$$

where $x_a = [1.7, 2]$, $x_b = [2, 1.5]$ and $x_c = [3.5, 1]$. It is important to note that the global minimizer of $\sum_{i=1}^{6} f_i(x_i)$ is approximately $x^\star \approx [2.47, 1.45]$. The communication topology is illustrated in Fig. 1.

The proposed controller is applied with control parameters set as $k_1 = 3$, $k_2 = 2$, $t_1 = T_c = 0.5s$, $T_p = 1s$, $\beta = 0.2$ and $h = 1.1$. The initial values of x_i are chosen as follows: $x_1 = [1, -3]$, $x_2 = [1.2, -1.2]$, $x_3 = [1.8, 1.6]$, $x_4 = [-1.2, 1.1]$, $x_5 = [2.7, 2.6]$ and $x_6 = [-1, -0.8]$. The initial conditions for $w_{i0}(t)$, $w_{i1}(t)$ and $k_{ij}(t)$ are randomly selected from the ranges $[0, 1]$, $[1, 5]$ and $[5, 10]$, respectively.

The simulation results obtained using the proposed control are presented in Figs. 2 and 3. From Fig. 2, it is evident that consensus within the multi-agent system is achieved at $t = T_c = 0.5s$, and the states of all agents converge to the minimizer $x^\star$ at $t = T_p = 1s$. It is worth noting that the entire optimization process can be divided into two stages: In the first time interval $[0, 0.5s)$, all agents estimate the gradient of the global objective function, as illustrated in Fig. 3. Simultaneously, the consensus control scheme facilitates achieving consensus among all agents at $t = 0.5$ s. In the second time interval $[0.5s, +\infty)$, once consensus has been reached and each agent has estimated the information of the global objective function, all agents' states converge to the global minimizer $x^\star$ at $t = T_p = 1s$.

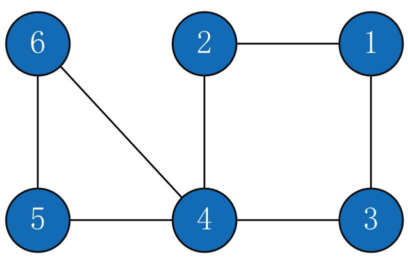
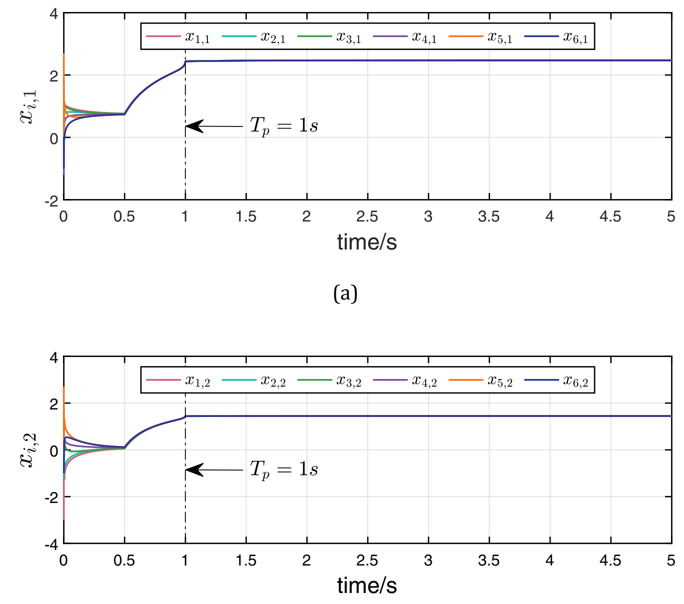


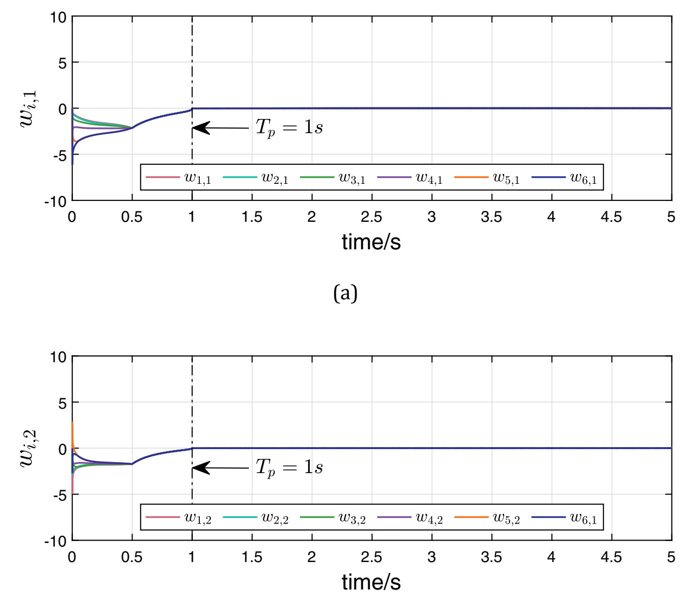
Fig. 1. Communication topology.



(a)

(b)

Fig. 2. State trajectories of the MAS with prescribed time $T_p = 1s$.



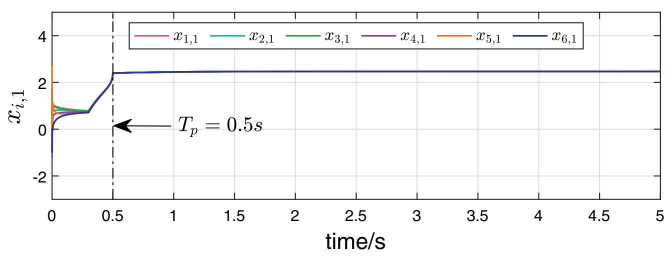
(a)

(b)

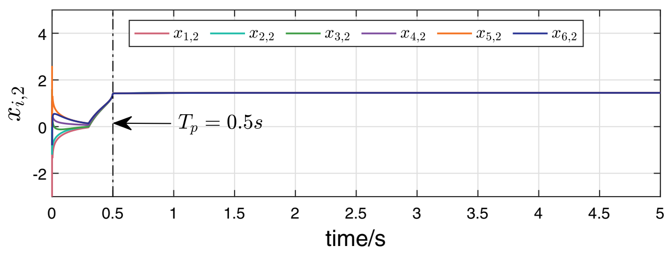
Fig. 3. Estimation of group gradients with $T_p = 1s$.

If the convergence times are set as $T_c = 0.3s$ and $T_p = 0.5s$, the simulation results are presented in Fig. 4.

From Figs. 2 to 4, it is evident that all the agents in the MAS successfully converge to the minimizer of the global objective function within a predefined time, thus the effectiveness of the proposed control scheme is validated.



(a)



(b)

Fig. 4. State trajectories of the MAS with prescribed time $T_p = 0.5s$.

5. Conclusion

In this paper, we presented a fully distributed optimization control strategy for first-order MAS that guarantees convergence within a prescribed time frame. This control scheme offers two advantages over conventional methods. First, the convergence time is independent of the initial conditions of the agents, allowing for explicit control through a designated time parameter. This characteristic is particularly critical in time-sensitive applications where predictability in system behavior is paramount. Second, the control scheme is fully distributed, meaning that the parameters of the controllers do not rely on global communication topology information.

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