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Leader-Following Attitude Consensus of Multiple Rigid Body Systems with an Uncertain Leader Over Jointly Connected Switching Networks

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Existing results on the leader-following consensus problem for multiple uncertain rigid body systems with an uncertain leader are limited to acyclic switching networks and require full availability of the leader's angular velocity states. In this paper, by leveraging an adaptive distributed observer for the uncertain leader system, we solve the leader-following consensus problem for multiple rigid body systems with an uncertain leader over jointly connected switching networks, which may be disconnected at any instant. Furthermore, the proposed control law depends only on the output of the leader system for the angular velocity and is robust to bounded external disturbances with unknown bounds. A numerical example is provided to demonstrate the effectiveness of the proposed design.

Keywords: Attitude consensus; rigid body systems; uncertain leader systems; switching graphs; disturbance rejection.

1. Introduction

The attitude consensus problem for multi-agent rigid body systems has been widely studied. This problem can be classified into two categories. In leaderless consensus, the objective is to design control laws such that the attitudes of all rigid body agents asymptotically synchronize with one another [1–5]. In contrast, leader-following consensus requires all agents' attitudes to converge to a desired trajectory generated by a leader system [6–9]. The leader-following consensus problem is considered more challenging and practically relevant because it allows explicit specification of the collective asymptotic behavior.

Early solutions to the leader-follower consensus problem of multiple rigid body systems, such as those in [10] and [11], relied on each follower having direct access to the leader's state, which resulted in a centralized control law. A distributed control law, however, permits each follower to use only its own information and that of its immediate neighbors. Various attempts have been made to design distributed control laws to achieve the leader-follower consensus for multiple rigid body systems in, for example, [6–9].

An effective framework for designing distributed control laws in multi-agent cooperative control is the distributed observer approach, first introduced in [12] for the cooperative output regulation problem. This approach proceeds in three steps: (i) construct a distributed observer that estimates the leader's state (and possibly its system matrix) over the communication network; (ii) design a local control law for each follower assuming full knowledge of the leader; (iii) replace the leader's information in the local control law with its distributed estimate. Since 2012, distributed

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observers have evolved through three main stages. In the first stage, the distributed observer only estimated all states of the leader, and assumed that all followers could access the dynamics of the leader. In the second stage, the distributed observer was enhanced to be able to estimate both the states and dynamics of the leader. Such a distributed observer is called the adaptive distributed observer. In the third stage, efforts have been made to further render the distributed observer the capability of dealing with an uncertain leader system. Such a distributed observer is called the adaptive distributed observer for an uncertain leader system. A comprehensive survey of these developments can be found in [13], and a most recent result on the adaptive distributed observer for an uncertain leader system was given in [14].

Progress on the leader-following attitude consensus problem for rigid body systems has closely paralleled advances in distributed observers. The first distributed solution appeared in [7], but assumed a known leader with known dynamics and a static, connected communication graph. Subsequent works progressively relaxed these assumptions by leveraging more sophisticated observers. For example, [9] studied the leader-following consensus problem of multiple uncertain rigid body systems over jointly connected networks by utilizing adaptive distributed observer. For the scenario similar to that of [9], Ref. [15] further considered the disturbance rejection problem for any bounded disturbances with unknown bounds. Reference [16] studied the leader-following consensus problem of multiple uncertain rigid body systems for an uncertain leader over acyclic and every time connected switching networks. Further details are provided in [13, Chap. 6].

Despite these advances, existing results still exhibit two notable limitations: (i) they require undirected communication among followers, and (ii) they impose restrictive topological conditions (e.g. acyclic connected switching graphs). To overcome these limitations, this paper employs the recent adaptive distributed observer proposed in [14], which relies solely on the leader's output, to address our problem. The resulting control law solves the leader-following attitude consensus problem with disturbance rejection for multiple uncertain rigid body systems and offers the following key advantages:

(1) It handles a broader class of uncertain leader systems.
(2) It operates over general jointly connected switching networks (without requiring acyclicity).
(3) It rejects arbitrary bounded disturbances with unknown bounds.

This paper is arranged as follows. Section 2 presents preliminaries and the problem formulation. In Sec. 3, we design an adaptive-distributed-observer-based control law and establish its effectiveness for the leader-following attitude consensus problem with disturbance rejection. Section 4 provides a numerical example to validate the proposed approach. Conclusions are drawn in Sec. 5.

Notation and Terminology. $\otimes$ denotes the Kronecker product of matrices. 1_N denotes an N dimensional column vector whose components are all 1. $\mathbf{0}$ represents a zero vector with compatible dimensions. $\|x\|$ denotes the Euclidean norm of vector x and $\|A\|$ denotes the induced 2-norm of matrix A. For $x_i \in R^{n_i}, i = 1, \dots, m$, $\mathrm{col}(x_1, \dots, x_m) = [x_1^T, \dots, x_m^T]^T$. For $x = \mathrm{col}(x_1, x_2, x_3) \in \mathbb{R}^3$, $x^\times$ is defined by

$$x^\times = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix},$$

which satisfies $x^\times x^T = \mathbf{0}$.

A quaternion $q \in \mathbb{R}^4$ is expressed by $q = \mathrm{col}(\hat{q}, \bar{q})$, where $\hat{q} \in \mathbb{R}^3$ is the vector part and $\bar{q} \in \mathbb{R}$ is the scalar part. The quaternion set is denoted by $\mathbb{Q}$, and the quaternion identity is denoted by $q_I = \mathrm{col}(0, 0, 0, 1) \in \mathbb{Q}$. Besides, the conjugate of q is denoted by $q^* = \mathrm{col}(-\hat{q}, \bar{q})$, and the product of two quaternions is denoted by $q_i \odot q_j = \begin{bmatrix} \bar{q}_i\hat{q}_j + \bar{q}_j\hat{q}_i + \hat{q}_i^\times\hat{q}_j \\ \bar{q}_i\bar{q}_j - \hat{q}_i^T\hat{q}_j \end{bmatrix}$, where $q, q_i, q_j \in \mathbb{Q}$. It can also be verified that $q \odot q_I = q_I \odot q = q$ and $(q_i \odot q_j)^* = q_j^* \odot q_i^*$. It is also noted that the quaternion multiplication does not satisfy the commutative rule, but satisfies the distributive and associative rules. Moreover, a quaternion $q \in \mathbb{Q}$ is a unit quaternion if it satisfies $\|q\| = 1$, and the set of unit quaternions is denoted by $\mathbb{Q}_u$. When q is a unit quaternion, it can be verified that its inverse $q^{-1} = q^*$.

Some operators are defined as follows:

$$\mathbf{Q}(x) = \mathrm{col}(x, 0), \quad x \in \mathbb{R}^3, \tag{1a}$$

$$\mathbf{C}(q) = (\bar{q}^2 - \hat{q}^T\hat{q})I_3 + 2\hat{q}\hat{q}^T - 2\bar{q}\hat{q}^\times, \quad q \in \mathbb{Q}, \tag{1b}$$

$$\mathbf{L}(x) = \begin{bmatrix} x_1 & 0 & 0 & 0 & x_3 & x_2 \\ 0 & x_2 & 0 & x_3 & 0 & x_1 \\ 0 & 0 & x_3 & x_2 & x_1 & 0 \end{bmatrix}$$

$$\text{for } x = \mathrm{col}(x_1, x_2, x_3), \tag{1c}$$

$$\mathrm{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases} \quad \text{for any } x \in \mathbb{R}. \tag{1d}$$

A function $\sigma : [0, \infty) \to \mathbb{R}^n$ is said to be piecewise continuous if it has at most countable discontinuous points denoted by $t_0, t_1, t_2, \dots$, satisfying $t_{i+1} - t_i \geq \tau$ for some dwell time $\tau > 0$ and $i = 0, 1, \dots$, and is continuous from the right at discontinuous points. A special case of a piecewise continuous function is the piecewise constant function $\sigma : [0, \infty) \to \mathcal{P} = \{1, 2, \dots, n_0\}$ where n_0 is some positive integer such that, for $i = 1, \dots, N$, $t_i \leq t < t_{i+1}$ with $t_0 = 0$, $\sigma(t) = r$ for some $r \in \mathcal{P}$.

2. Preliminaries and Problem Formulation

Like in [7] and [15], we describe the attitude kinematics and dynamics of multiple rigid-body systems as follows:

$$\dot{q}_i = \frac{1}{2} q_i \odot \mathbf{Q}(\omega_i), \tag{2a}$$

$$J_i \dot{\omega}_i = -\omega_i^{\times} J_i \omega_i + u_i + d_i, \tag{2b}$$

where $i = 1, \ldots, N$; $q_i \in \mathbb{Q}_u$ is the attitude of the body frame relative to the inertial frame represented by unit quaternion; $\omega_i \in \mathbb{R}^3$ is the angular velocity of the body frame relative to the inertial frame;$u_i \in \mathbb{R}^3$ denotes the control input of the ith rigid body system; $d_i \in \mathbb{R}^3$ denotes the external disturbances; $J_i \in \mathbb{R}^{3\times 3}$ denotes the inertial matrix of the ith rigid-body system and is positive definite. It should be pointed out that q_i, ω_i, u_i, d_i, and J_i are all expressed in the body frame, and the external disturbances d_i satisfy the following assumption:

Assumption 1. For $i = 1, \ldots, N$, $d_i : [0, \infty) \to \mathbb{R}^3$ is a piecewise continuous bounded time-varying function with the bound unknown, i.e. $\|d_i(t)\|_\infty \leq D_i$ for some unknown positive number D_i.

The target system relative to the inertial frame is expressed as follows:

$$\dot{q}_0 = \frac{1}{2} q_0 \odot \mathbf{Q}(\omega_0), \tag{3a}$$

$$\omega_0 = \operatorname{col}(\omega_{01}, \omega_{02}, \omega_{03}), \tag{3b}$$

$$\omega_{0k} = \sum_{i=1}^{n_k} \Omega_{ki} \sin(\psi_{0ki} t + \sigma_{ki}), \quad k = 1, 2, 3, \tag{3c}$$

where $n_k \in \mathbb{N}^+$, Ω_{ki} and ψ_{0ki} are positive unknown constants, $0 \leq \sigma_{ki} < \pi$, $q_0 \in \mathbb{Q}_u$ and $\omega_0 \in \mathbb{R}^3$ are the desired attitude and angular velocity of the target system's fixed body frame relative to the inertial frame, and ω_{0k} represents the component of ω_0 on the axis k. As in [7] and [15], we view system (2) and (3) as a multi-agent system with $N+1$ agents. Subsystem (3) represents the leader, and subsystems (2) represent N followers. Let $\sigma(t)$ be a piecewise constant signal. Then, the communication network for systems (2) and (3) is described by a time-varying digraph $\mathcal{G}_{\sigma(t)} = (\mathcal{V}, \mathcal{E}_{\sigma(t)})$ with $\mathcal{V} = \{0, 1, \ldots, N\}$ and $\mathcal{E}_{\sigma(t)} \subseteq \mathcal{V} \times \mathcal{V}$ for all $t \geq 0$. Node 0 is associated with the leader system (14) and nodes $i, i = 1, \ldots, N$, are associated with the ith subsystem of system (2). For $i = 0, 1, \ldots, N, j = 1, \ldots, N, i \neq j, (i, j) \in \mathcal{E}_{\sigma(t)}$ if and only if u_j can use the information of agent i for control at time instant t. The neighbor set of agent i at time t is denoted by $\mathcal{N}_i(t) = \{j : (j, i) \in \mathcal{E}_{\sigma(t)}\}$. The weighted adjacency matrix of a graph $\mathcal{G}_{\sigma(t)}$ is denoted by a nonnegative matrix $\mathcal{A}_{\sigma(t)} = [a_{ij}(t)]_{i,j=0}^N \in \mathbb{R}^{(N+1)\times(N+1)}$ where $a_{ii}(t) = 0$. For $i, j = 0, 1, 2, \ldots, N$ and $i \neq j$, $a_{ij}(t) > 0$ if and only if $(j, i) \in \mathcal{E}_{\sigma(t)}$ and $a_{ij}(t) = 0$ if otherwise.

We assume the digraph satisfies the following so-called jointly connected condition.

Assumption 2. There exists a subsequence $\{j_k : k = 0, 1, 2, \ldots\}$ of $\{j : j = 0, 1, 2, \ldots\}$ with $t_{j_0} = 0$ and $t_{j_{k+1}} - t_{j_k} \leq T_c$ for some positive constant T_c, such that the union graph $\bigcup_{t_j \in [t_{j_k}, t_{j_{k+1}})} \mathcal{G}_{\sigma(t_j)}$ contains a spanning tree with node 0 as the root.

According to [9, 17, 18], the errors of attitude and the angular velocity of followers relative to the leader are defined as follows:

$$q_{ei} = q_0^{-1} \odot q_i, \tag{4a}$$

$$\omega_{ei} = \omega_i - \mathbf{C}(q_{ei})\omega_0, \tag{4b}$$

where $q_{ei} \in \mathbb{Q}_u$ and $\omega_{ei} \in \mathbb{R}^3$. Then the error dynamics is as follows [7] and [15]:

$$\dot{q}_{ei} = \frac{1}{2} q_{ei} \odot \mathbf{Q}(\omega_{ei}), \tag{5a}$$

$$\begin{aligned} J_i \dot{\omega}_{ei} = {} & -\omega_i^{\times} J_i \omega_i + J_i(\omega_{ei}^{\times} \mathbf{C}(q_{ei})\omega_0 \\ & - \mathbf{C}(q_{ei})\dot{\omega}_0) + u_i + d_i. \end{aligned} \tag{5b}$$

We consider the distributed control law of the following form: for $i = 1, \ldots, N$,

$$u_i = f_i(q_i, \omega_i, z_i, \varphi_i), \tag{6a}$$

$$\dot{\varphi}_i = g_i(\varphi_i, \varphi_j, j \in \mathcal{N}_i(t)), \tag{6b}$$

$$\dot{z}_i = h_i(q_i, \omega_i, z_i, \varphi_i), \tag{6c}$$

where φ_i is the state of the distributed observer that will be specified later, z_i is the state of the adaptive control law, and $f_i(\cdot), g_i(\cdot), h_i(\cdot)$ are globally defined nonlinear functions to be designed.

The leader-following consensus with disturbance rejection problem for multiple uncertain rigid body systems is described as follows:

Problem 1. *Given systems* (2), (3), *and a switching communication graph $\mathcal{G}_{\sigma(t)}$, design a distributed state feedback control law of the form* (6) *such that, for any initial condition $q_0(0) \in \mathbb{Q}_u$, any $q_i(0) \in \mathbb{Q}_u$, any $\omega_i(0)$, and any $\varphi_i(0), i = 1, \ldots, N$, the solution of the closed-loop system exists and satisfies*

$$\lim_{t\to\infty} \hat{q}_{ei}(t) = 0, \quad \lim_{t\to\infty} \omega_{ei}(t) = 0 \tag{7}$$

for $i = 1, \ldots, N$.

3. Main Result

In this section, we will present our main result. For this purpose, we first construct an exponentially convergent estimator for the uncertain angular velocities (3c).

3.1. *Exponentially convergent estimator*

To deal with the uncertain angular velocities (3c), we need to make use of the adaptive distributed observer for unknown leader system developed in [14]. For this purpose, we will derive an exosystem that can produce the unknown angular velocities ω_{0k}, $k = 1, 2, 3$. Let $\theta_{0k} = \mathrm{col}(\theta_{0k1}, \ldots, \theta_{0kn_k})$, $k = 1, 2, 3$, be such that

$$\prod_{i=1}^{n_k}(s^2 + \psi_{0ki}^2) = s^{2n_k} + \theta_{0k1}s^{2(n_k-1)} + \cdots + \theta_{0kn_k}. \quad (8)$$

Then, it can be verified that, for $k = 1, 2, 3$, $\omega_{0k}(t)$ satisfies the following equation:

$$\frac{d^{2n_k}\omega_{0k}(t)}{dt^{2n_k}} + \theta_{0k1}\frac{d^{2(n_k-1)}\omega_{0k}(t)}{dt^{2(n_k-1)}} + \cdots + \theta_{0k(n_k-1)} \times \frac{d^2\omega_{0k}(t)}{dt^2} + \theta_{0kn_k}\omega_{0k} = 0, \quad k = 1, 2, 3. \quad (9)$$

For $k = 1, 2, 3$, let

$$\varrho_k = \left[\omega_{0k} \ \frac{d\omega_{0k}(t)}{dt} \ \frac{d^2\omega_{0k}(t)}{dt^2} \cdots \frac{d^{2n_k-1}\omega_{0k}(t)}{dt^{2n_k-1}}\right]^T \quad (10)$$

and

$$P_k = \begin{bmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ \theta_{0k1} & 0 & 1 & & & & & \\ 0 & \theta_{0k1} & 0 & \ddots & & & & \\ & & \theta_{0k1} & \ddots & \ddots & & & \\ 0 & & & \ddots & \ddots & 1 & & \\ \theta_{0k(n_k-1)} & 0 & & & \ddots & 0 & 1 & \\ 0 & \theta_{0k(n_k-1)} & 0 & & & \theta_{0k1} & 0 & 1 \end{bmatrix}. \quad (11)$$

Then, it can be verified that ω_{0k} can be generated by the following autonomous system:

$$\dot{\zeta}_k(t) = S_k(\theta_{0k})\zeta_k(t), \quad \omega_{0k} = \mathbf{c}_k\zeta_k(t), \ \ k = 1, 2, 3, \quad (12)$$

where the pair $(\mathbf{c}_k, S_k(\theta_{0k}))$ is in the following observable canonical form:

$$S_k(\theta_{0k}) = A - \mathrm{col}(0, \theta_{0k1}, 0, \theta_{0k2}, \ldots, 0, \theta_{0kn_k})\mathbf{c}_k, \quad (13a)$$

$$\mathbf{c}_k = \begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix} \in \mathbb{R}^{1\times 2n_k}, \quad (13b)$$

$$A = \begin{bmatrix}\mathbf{0} & I_{2n_k-1} \\ 0 & \mathbf{0}\end{bmatrix} \in \mathbb{R}^{2n_k\times 2n_k}. \quad (13c)$$

Now we can define the following leader system:

$$\dot{q}_0 = \frac{1}{2}q_0 \odot \mathbf{Q}(\omega_0), \quad (14a)$$

$$\omega_0 = \mathrm{col}(\omega_{01}, \omega_{02}, \omega_{03}), \quad (14b)$$

$$\dot{\zeta}_k(t) = S_k(\theta_{0k})\zeta_k(t), \quad (14c)$$

$$\omega_{0k} = \mathbf{c}_k\zeta_k(t), \quad k = 1, 2, 3, \quad (14d)$$

which is in the same form as that in [7] and [15] except that the matrices $S_k(\theta_{0k})$ contain unknown parameters.

Remark 3.1. The relationship between the initial conditions of ζ_k and $\varrho_k(0)$ is as follows:

$$\zeta_k(0) = P_k\varrho_k(0), \quad k = 1, 2, 3, \quad (15)$$

where

$$\varrho_k(0) = \begin{bmatrix} \sum_{i=1}^{n_k}\Omega_{ki}\sin(\sigma_{ki}) \\ \sum_{i=1}^{n_k}\Omega_{ki}\psi_{0ki}\cos(\sigma_{ki}) \\ \sum_{i=1}^{n_k}-\Omega_{ki}\psi_{0ki}^2\sin(\sigma_{ki}) \\ \cdots \\ \sum_{i=1}^{n_k}(-1)^{n_k-1}\Omega_{ki}\psi_{0ki}^{2n_k-1}\cos(\sigma_{ki}) \end{bmatrix}. \quad (16)$$

Since the pair $(\mathbf{c}_k, S_k(\theta_{0k}))$ is in the standard observable canonical form, with $y_k := \omega_{0k}$, $k = 1, 2, 3$, we can directly apply the exponentially convergent estimator for the leader system (12) given by [14] as follows: for $k = 1, 2, 3$,

$$\dot{\eta}_k(t) = \mathbf{D}_k\eta_k(t) + \mathbf{d}_ky_k(t), \quad (17a)$$

$$\dot{\xi}_k(t) = \mathbf{D}_k\xi_k(t) - \mathbf{b}_ky_k(t), \quad (17b)$$

$$\dot{\hat{\theta}}_{0k}(t) = -\gamma_k\phi_k(t)(\hat{y}_k(t) - y_k(t)), \quad (17c)$$

$$\hat{v}_k(t) = \eta_k(t) + \Xi_k(t)\hat{\theta}_{0k}(t), \quad (17d)$$

where $\eta_k(t) \in \mathbb{R}^{2n_k}, \xi_k(t) \in \mathbb{R}^{2n_k}, \hat{\theta}_{0k}(t) \in \mathbb{R}^{n_k}, \hat{v}_k(t) \in \mathbb{R}^{2n_k}$,

$$\mathbf{b}_k = [0 \quad \cdots \quad 0 \quad 1]^T \in \mathbb{R}^{1\times 2n_k}, \quad (18)$$

$\mathbf{d}_k \in \mathbb{R}^{2n_k}$ is such that the matrix

$$\mathbf{D}_k := A - \mathbf{d}_k\mathbf{c}_k \in \mathbb{R}^{2n_k\times 2n_k} \quad (19)$$

is Hurwitz, γ_k is any positive constant, $\Xi_k(t) \in \mathbb{R}^{2n_k\times n_k}$ is defined as

$$\Xi_k(t) := \left[\mathbf{D}_k^{2(n_k-1)}\xi_k(t)\,\mathbf{D}_k^{2(n_k-2)}\xi_k(t) \cdots \mathbf{D}_k^2\xi_k(t)\,\xi_k(t)\right], \quad (20)$$

$\phi_k(t) \in \mathbb{R}^{n_k}$ is defined as $\phi_k(t) := (\mathbf{c}_k \Xi_k(t))^T$, and $\hat{y}_k(t) \in \mathbb{R}$ is defined as

$$\hat{y}_k(t) := \mathbf{c}_k \eta_k(t) + \phi_k^T(t) \hat{\theta}_{0k}(t). \tag{21}$$

System (17) is called the exponentially convergent estimator for the leader system (12) because of the following result:

Lemma 3.1 ([14, Proposition 1]). *Suppose that* $\zeta_k(0)$, $k = 1, 2, 3$, *given by* (14c) *is such that none of* Ω_{ki}, $i = 1, \dots, n_k$, *in* (3c) *is equal to zero. Then, for any* $(\eta_k(0), \xi_k(0), \hat{\theta}_{0k}(0)) \in \mathbb{R}^{2n_k} \times \mathbb{R}^{2n_k} \times \mathbb{R}^{n_k}$, $k = 1, 2, 3$, *the solution of* (17) *exists over* $[0, \infty)$ *and satisfies*

$$\lim_{t\to\infty} (\hat{\theta}_{0k}(t) - \theta_{0k}) = \mathbf{0}, \tag{22a}$$

$$\lim_{t\to\infty} (\hat{v}_k(t) - \zeta_k(t)) = \mathbf{0} \tag{22b}$$

both exponentially.

Remark 3.2. Similar to [14, Remark 5], the assumption that $\zeta_k(0)$ is such that none of Ω_{ki}, $k = 1, 2, 3$, $i = 1, \dots, n_k$, in (3c) is equal to zero is not material. In fact, if, for some j, $\Omega_{kj} = 0$, then the mode corresponding to Ω_{kj} is not excited. In this case, we only need to remove the components $\Omega_{kj} \sin(\psi_{0kj} t + \sigma_{kj})$ in (3c).

Based on [14, Lemma 3.1] presented the following distributed dynamic compensator: for $i = 1, \dots, N$, and $k = 1, 2, 3$,

$$\dot{\hat{\theta}}_{ik}(t) = \mu_{1k} \sum_{j=0}^{N} a_{ij}(t)(\hat{\theta}_{jk}(t) - \hat{\theta}_{ik}(t)), \tag{23a}$$

$$\dot{v}_{ik}(t) = S_k(\hat{\theta}_{ik}(t)) v_{ik}(t) + \mu_{2k} \sum_{j=0}^{N} a_{ij}(t) \times (v_{jk}(t) - v_{ik}(t)), \tag{23b}$$

where $\hat{\theta}_{ik}(t) = \mathrm{col}(\hat{\theta}_{ik1}(t), \dots, \hat{\theta}_{ikn_k}(t)) \in \mathbb{R}^{n_k}$, $v_{ik}(t) \in \mathbb{R}^{2n_k}$, $v_{0k}(t) = \hat{v}_k(t)$, and μ_{1k} and μ_{2k} are positive constants.

Then, we have the following result:

Lemma 3.2 ([14, Theorem 1]). *Under Assumption* 2, *for any* $\zeta_k(0)$ *such that none of* Ω_{ki}, $k = 1, 2, 3$, $i = 1, \dots, n_k$, *in* (3c) *is equal to zero, and for any* $(\eta_k(0), \xi_k(0), \hat{\theta}_{0k}(0)) \in \mathbb{R}^{2n_k} \times \mathbb{R}^{2n_k} \times \mathbb{R}^{n_k}$, *and any* $(\hat{\theta}_{ik}(0), v_{ik}(0)) \in \mathbb{R}^{n_k} \times \mathbb{R}^{2n_k}$, $i = 1, \dots, N$, $k = 1, 2, 3$, *the solutions of the dynamic compensator* (23) *exist over* $[0, \infty)$ *and satisfy*

$$\lim_{t\to\infty} (\hat{\theta}_{ik}(t) - \theta_{0k}) = \mathbf{0}, \tag{24a}$$

$$\lim_{t\to\infty} (v_{ik}(t) - \zeta_k(t)) = \mathbf{0} \tag{24b}$$

both exponentially.

3.2. *The output-based adaptive distributed observer for multiple rigid body systems*

To construct an adaptive distributed observer to estimate the known leader (17) for the multiple rigid body systems, we further present a dynamic compensator as follows: for $i = 1, \dots, N$, $k = 1, 2, 3$,

$$\dot{\Phi}_i = \frac{1}{2} \Phi_i \odot \mathbf{Q}(\Pi_i) + \mu_\Phi \sum_{j=0}^{N} a_{ij}(t)(\Phi_j - \Phi_i), \tag{25a}$$

$$\dot{F}_{ik} = \mu_{Fk} \sum_{j=0}^{N} a_{ij}(t)(F_{jk} - F_{ik}), \tag{25b}$$

where $\Phi_i \in \mathbb{Q}$, $F_{ik} \in \mathbb{R}^{1\times 2n_k}$, $\Phi_0 = q_0$, $F_{0k} = \mathbf{c}_k$, μ_{Fk} and μ_Φ are positive constants. Also, let

$$F_i = \begin{bmatrix} F_{i1} & & \\ & F_{i2} & \\ & & F_{i3} \end{bmatrix} \in \mathbb{R}^{3\times 2\bar{n}},$$

$$F_0 = \begin{bmatrix} \mathbf{c}_1 & & \\ & \mathbf{c}_2 & \\ & & \mathbf{c}_3 \end{bmatrix} \in \mathbb{R}^{3\times 2\bar{n}}, \tag{26}$$

$$v_i = \begin{bmatrix} v_{i1} \\ v_{i2} \\ v_{i3} \end{bmatrix} \in \mathbb{R}^{2\bar{n}}$$

with $\bar{n} = n_1 + n_2 + n_3$, and

$$\Pi_i = F_i v_i. \tag{27}$$

Lemma 3.3. *Under Assumption* 2, *for any initial conditions* $(\Phi_i(0), \Pi_i(0), F_{ik}(0)) \in \mathbb{Q} \times \mathbb{R}^3 \times \mathbb{R}^{1\times 2n_k}$, $i = 1, \dots, N$, $k = 1, 2, 3$, *the solutions of the dynamic compensator* (25) *satisfy*

$$\lim_{t\to\infty} (F_{ik}(t) - F_{0k}(t)) = \mathbf{0}_{1\times 2n_k}, \tag{28a}$$

$$\lim_{t\to\infty} (\Pi_i(t) - \omega_0(t)) = \mathbf{0}, \tag{28b}$$

$$\lim_{t\to\infty} (\Phi_i(t) - q_0(t)) = \mathbf{0} \tag{28c}$$

exponentially.

Proof. Let $\tilde{F}_{ik}(t) = F_{ik}(t) - F_{0k}(t)$ and denote the following compact form:

$$\tilde{F}_i = \begin{bmatrix} \tilde{F}_{i1} & & \\ & \tilde{F}_{i2} & \\ & & \tilde{F}_{i3} \end{bmatrix} \in \mathbb{R}^{3\times 2\bar{n}}, \quad \zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} \in \mathbb{R}^{2\bar{n}},$$

$$\theta_0 = \begin{bmatrix} \theta_{01} \\ \theta_{02} \\ \theta_{03} \end{bmatrix} \in \mathbb{R}^{\bar{n}}, \quad \hat{\theta}_i = \begin{bmatrix} \hat{\theta}_{i1} \\ \hat{\theta}_{i2} \\ \hat{\theta}_{i3} \end{bmatrix} \in \mathbb{R}^{\bar{n}},$$

$$S(\theta_0) = \begin{bmatrix} S_1(\theta_{01}) & & \\ & S_2(\theta_{02}) & \\ & & S_3(\theta_{03}) \end{bmatrix} \in \mathbb{R}^{2\bar{n}\times 2\bar{n}},$$

$$S(\hat{\theta}_i) = \begin{bmatrix} S_1(\hat{\theta}_{i1}) & & \\ & S_2(\hat{\theta}_{i2}) & \\ & & S_3(\hat{\theta}_{i3}) \end{bmatrix} \in \mathbb{R}^{2\bar{n}\times 2\bar{n}}. \tag{29}$$

Then (28a) can be directly obtained from [13, Theorem 4.8]. On the other hand, we have

$$\Pi_i - \omega_0 = F_i v_i - F_0\zeta. \tag{30}$$

Thus, (28a) and (24b) imply (28b). Finally, by [13, Corollary 6.1], (28b) implies (28c). □

Next, let

$$\Pi_i^d = F_i S(\hat{\theta}_i) v_i. \tag{31}$$

Then,

$$\Pi_i^d - \dot{\omega}_0 = F_i S(\hat{\theta}_i) v_i - F_0 S(\theta_0)\zeta, \tag{32a}$$

$$\Pi_i^d - \dot{\Pi}_i = -\mu_F \sum_{j=0}^{N} a_{ij}(t)(F_j - F_i)v_i - \mu_2 F_i \sum_{j=0}^{N} a_{ij}(t)(v_j - v_i), \tag{32b}$$

where $\mu_F = \mathrm{diag}(\mu_{F1}, \mu_{F2}, \mu_{F3})$ and $\mu_2 = \mathrm{diag}(\mu_{21}, \mu_{22}, \mu_{23})$. Substituting (22) and (28a) into (32) gives the following corollary of Lemma 3.3.

Corollary 3.1. *Under Assumption* 2, *for any initial conditions* $(F_{ik}(0), v_i(0), \hat{\theta}_i(0))$, $i = 1, \dots, N$,

$$\lim_{t\to\infty}(\Pi_i^d(t) - \dot{\omega}_0(t)) = \mathbf{0}, \tag{33a}$$

$$\lim_{t\to\infty}(\Pi_i^d(t) - \dot{\Pi}_i(t)) = \mathbf{0}. \tag{33b}$$

Lemmas 3.2 and 3.3 imply the following result.

Theorem 3.1. *Under Assumption* 2, *for any* $\zeta_k(0)$ *such that none of* Ω_{ki}, $k = 1, 2, 3$, $i = 1, \dots, n_k$, *in* (3c) *is equal to zero, any* $(\eta_k(0), \xi_k(0), \hat{\theta}_{0k}(0)) \in \mathbb{R}^{2n_k} \times \mathbb{R}^{2n_k} \times \mathbb{R}^{n_k}$, *and any* $(\hat{\theta}_{ik}(0), v_{ik}(0), \Phi_i(0), F_{ik}(0)) \in \mathbb{R}^{n_k} \times \mathbb{R}^{2n_k} \times \mathbb{Q} \times \mathbb{R}^{1\times 2n_k}$, $i = 1, \dots, N$, $k = 1, 2, 3$, *the solutions of the adaptive distributed observer* (23) *and* (25) *exist over* $[0, \infty)$ *and satisfy* (24) *and* (28), *respectively.*

For this reason, we call the composition of the two dynamic compensators (23) and (25) the output-based adaptive distributed observer for the multiple rigid body systems.

3.3. *Estimated error dynamics*

To obtain a distributed control law, like in [7], replacing q_0 and ω_0 in (4) by their estimates Φ_i and Π_i gives the following estimated tracking errors: for $i = 1, \dots, N$,

$$q_{\epsilon i} = \Phi_i^* \odot q_i, \tag{34a}$$

$$\omega_{\epsilon i} = \omega_i - \mathbf{C}(q_{\epsilon i})\Pi_i. \tag{34b}$$

Remark 3.3. As explained in [13, Remark 6.6], for $i = 1, \dots, N$, since $\|q_i(t)\| = 1$ for all $t \geq 0$, we have $\|q_{\epsilon i}(t)\| = \|\Phi_i^* \odot q_i(t)\| = \|\Phi_i^*\|\|q_i(t)\| = \|\Phi_i^*\|$ for all $t \geq 0$. Then, due to $\|q_0(t)\| = 1$ for all $t \geq 0$, we can obtain $\lim_{t\to\infty} \|q_{\epsilon i}(t)\| = 1$ exponentially from the fact that $\|q_{\epsilon i}(t)\| = \|\Phi_i^*\|$ for all $t \geq 0$ and equation (28c).

By [15, Remark 7], we have the following quite obvious result.

Lemma 3.4. *If the solutions of the adaptive distributed observer* (23) *and* (25) *satisfy* (24) *and* (28), *respectively, then* $\lim_{t\to\infty}(\hat{q}_{\epsilon i}(t) - \hat{q}_{ei}(t)) = \mathbf{0}$ *and* $\lim_{t\to\infty}(\omega_{\epsilon i}(t) - \omega_{ei}(t)) = \mathbf{0}$, *exponentially.*

Next, it can be verified that the dynamics of the estimated tracking errors is governed by the following system [15]:

$$\dot{q}_{\epsilon i} = \frac{1}{2} q_{\epsilon i} \odot \mathbf{Q}(\omega_{\epsilon i}) + M_{q\epsilon i}, \tag{35a}$$

$$J_i\dot{\omega}_{\epsilon i} = -\omega_i^{\times} J_i \omega_i + J_i \omega_{\epsilon i}^{\times} \mathbf{C}(q_{\epsilon i})\Pi_i - J_i \mathbf{C}(q_{\epsilon i})\Pi_i^d + u_i + d_i - M_{\omega\epsilon i}, \tag{35b}$$

where

$$M_{q\epsilon i} = \frac{1}{2}(q_{\epsilon i}^{\top} q_{\epsilon i} - 1)\mathbf{Q}(\Pi_i) \odot q_{\epsilon i} + \Delta_i^* \odot q_i, \tag{36a}$$

$$M_{C\epsilon i} = 2\bar{q}_{\epsilon i}\bar{M}_{q\epsilon i} I_3 - 2\hat{q}_{\epsilon i}^{\top}\hat{M}_{q\epsilon i} I_3 + 2\hat{M}_{q\epsilon i}\hat{q}_{\epsilon i}^{\top} + 2\hat{q}_{\epsilon i}\hat{M}_{q\epsilon i}^{\top} - 2\bar{M}_{q\epsilon i}\hat{q}_{\epsilon i}^{\times} - 2\bar{q}_{\epsilon i}\hat{M}_{q\epsilon i}^{\times}, \tag{36b}$$

$$M_{\omega\epsilon i} = J_i M_{C\epsilon i}\Pi_i + J_i \mathbf{C}(q_{\epsilon i})(\dot{\Pi}_i - \Pi_i^d) \tag{36c}$$

and

$$\Delta_i = \mu_\Phi \sum_{j=0}^{N} a_{ij}(t)(\Phi_j - \Phi_i), \tag{37}$$

which decays to zero exponentially from (28c).

Further, let

$$\Omega_{\epsilon i} = \omega_{\epsilon i} + k_{1i}\hat{q}_{\epsilon i}, \tag{38}$$

where $k_{1i} > 0$. Then (35) is transformed to the following:

$$\dot{q}_{\epsilon i} = \frac{1}{2} q_{\epsilon i} \odot \mathbf{Q}(\omega_{\epsilon i}) + M_{q\epsilon i}, \tag{39a}$$

$$J_i\dot{\Omega}_{\epsilon i} = J_i\bigg(\omega_{\epsilon i}^{\times}\mathbf{C}(q_{\epsilon i})\Pi_i - \mathbf{C}(q_{\epsilon i})\Pi_i^d + \frac{k_{1i}}{2}(\bar{q}_{\epsilon i}I_3 + \hat{q}_{\epsilon i}^{\times})\omega_{\epsilon i}\bigg) - \omega_i^{\times}J_i\omega_i + u_i + d_i - M_{\Omega\epsilon i}, \quad (39b)$$

where

$$M_{\Omega\epsilon i} = M_{\omega\epsilon i} - k_{1i}J_i\hat{M}_{q\epsilon i}. \quad (40)$$

Remark 3.4. It follows from Eq. (28b) that, for $i = 1, \dots, N$, $\Pi_i(t)$ are uniformly bounded over $[0, \infty)$, and from Remark 3.3 that $\lim_{t\to\infty}\|q_{\epsilon i}(t)\| = 1$ exponentially. Hence $M_{q\epsilon i}(t)$ are uniformly bounded and converge to zero exponentially. As a result $M_{C\epsilon i}(t), M_{\omega\epsilon i}(t)$, $M_{\Omega\epsilon i}(t)$ are all uniformly bounded over $[0, \infty)$ and decay to zero exponentially.

By [13, Lemma 6.2], system (39a) satisfies the following interesting property.

Lemma 3.5. *Consider system* (39a), *where* $q_{\epsilon i}(t) \in \mathbb{Q}$, *and* $\omega_{\epsilon i}(t) \in \mathbb{R}^3$ *and* $M_{q\epsilon i}(t) \in \mathbb{Q}$ *are both piecewise continuous in* t *and uniformly bounded over* $[0, \infty)$. *Suppose* $\lim_{t\to\infty}\|q_{\epsilon i}(t)\| = 1$ *and* $\lim_{t\to\infty} M_{q\epsilon i} = \mathbf{0}$. *Then if* (38) *satisfies* $\lim_{t\to\infty}\Omega_{\epsilon i}(t) = \mathbf{0}$, *the solution of* (39a) *is bounded and is such that* $\lim_{t\to\infty}\hat{q}_{\epsilon i}(t) = \mathbf{0}$ *and hence* $\lim_{t\to\infty}\omega_{\epsilon i}(t) = \mathbf{0}$.

As a result, in what follows, it suffices to stabilize (39b).

3.4. *Solution of Problem* 1

Eq. (39b) is an uncertain nonlinear system involving the uncertain inertia matrix J_i and the uncertain disturbance d_i. To handle J_i, like in [15], we introduce a linear operator $\mathbf{L}(\cdot)$ as defined in (1c). It can be verified that for any $x = \mathrm{col}(x_1, x_2, x_3) \in \mathbb{R}^3$, $J_i x = \mathbf{L}(x)\Theta_i$, where

$$\Theta_i = \mathrm{col}(J_{i_{11}}, J_{i_{22}}, J_{i_{33}}, J_{i_{23}}, J_{i_{13}}, J_{i_{12}}), \quad (41)$$

and $J_{i_{mn}}, m, n = 1, 2, 3$, are the mth row and nth column of J_i. Let

$$\chi_i = \omega_{\epsilon i}^{\times}\mathbf{C}(q_{\epsilon i})\Pi_i - \mathbf{C}(q_{\epsilon i})\Pi_i^d + \frac{k_{1i}}{2}(\bar{q}_{\epsilon i}I_3 + \hat{q}_{\epsilon i}^{\times})\omega_{\epsilon i}. \quad (42)$$

Then (39) can be written as follows:

$$\dot{q}_{\epsilon i} = \frac{1}{2}q_{\epsilon i} \odot \mathbf{Q}(\omega_{\epsilon i}) + M_{q\epsilon i}, \quad (43a)$$

$$J_i\dot{\Omega}_{\epsilon i} = (-\omega_i^{\times}\mathbf{L}(\omega_i) + \mathbf{L}(\chi_i))\Theta_i + u_i + d_i - M_{\Omega\epsilon i}. \quad (43b)$$

To handle the unknown external disturbances, let $\mathrm{sign}(x) = \mathrm{col}(\mathrm{sgn}(x_1), \mathrm{sgn}(x_2), \mathrm{sgn}(x_3))$ for any $x = \mathrm{col}(x_1, x_2, x_3) \in \mathbb{R}^3$.

Now, based on the estimation of q_0, ω_0, and $\dot{\omega}_0$, an estimation-based adaptive control law is given as follows:

$$u_i = -k_{2i}\Omega_{\epsilon i} + (\omega_i^{\times}\mathbf{L}(\omega_i) - \mathbf{L}(\chi_i))\hat{\Theta}_i - \hat{D}_i\mathrm{sign}(\Omega_{\epsilon i}), \quad (44a)$$

$$\dot{\hat{\Theta}}_i = \Lambda_i(-\omega_i^{\times}\mathbf{L}(\omega_i) + \mathbf{L}(\chi_i))^T\Omega_{\epsilon i}, \quad (44b)$$

$$\dot{\hat{D}}_i = \mathrm{sign}^T(\Omega_{\epsilon i})\Omega_{\epsilon i}, \quad i = 1, \dots, N, \quad (44c)$$

where $k_{2i} > 0$, $\Lambda_i \in \mathbb{R}^{6\times 6}$ are positive definite matrices, $\hat{\Theta}_i \in \mathbb{R}^6$ are the estimates of Θ_i, and $\hat{D}_i \in \mathbb{R}$ are the estimates of D_i.

To deal with the discontinuity caused by the switching graph $\mathcal{G}_{\sigma(t)}$ and the piecewise continuous disturbances $d_i(t)$, let us note that, by the definition of the piecewise continuous/constant function, the union $\mathcal{U}$ of the switching instants of the switching signal $\sigma(t)$ and all discontinuous points of $d_i(t)$ is countable. Denote the elements of $\mathcal{U}$ by $\mathcal{U} = \{\tau_0, \tau_1, \tau_2, \dots\}$ such that $\tau_0 = 0, \tau_i < \tau_{i+1}, i = 0, 1, \dots,$. Then, we need one more assumption as follows:

Assumption 3. There exists a positive number τ such that $\tau_{j+1} - \tau_j \geq \tau$ for all $j = 0, 1, 2, \dots$.

Remark 3.5. Assumption 3 is also used in [19] which deals with the consensus with disturbance rejection problem of multiple Euler–Lagrange systems. As pointed out in [19], Assumption 3 is not restrictive, because it is satisfied automatically if the switching signal $\sigma(t)$ is periodic and all d_i have finitely many discontinuities or are periodic with infinitely many discontinuities.

Then we have the following main result.

Theorem 3.2. *Under Assumptions* 1 *to* 3, *for any* $\zeta_k(0)$ *such that none of* Ω_{ki}, $k = 1, 2, 3$, $i = 1, \dots, n_k$, *in* (3c) *is equal to zero, any* $(\eta_k(0), \xi_k(0), \hat{\theta}_{0k}(0)) \in \mathbb{R}^{2n_k} \times \mathbb{R}^{2n_k} \times \mathbb{R}^{n_k}$, *and any* $(\Phi_i(0), \hat{\theta}_{ik}(0), F_{ik}(0), v_{ik}(0)) \in \mathbb{Q} \times \mathbb{R}^{n_k} \times \mathbb{R}^{1\times 2n_k} \times \mathbb{R}^{2n_k}$, $i = 1, \dots, N, k = 1, 2, 3$, *Problem* 1 *is solvable by the distributed control law composed of* (44), (17), (23), *and* (25).

Proof. By Theorem 3.1, (24), (28), and (33) are all satisfied, which implies all conditions of Lemmas 3.4 and 3.5 are satisfied. Thus we can directly use these two lemmas in the following proof.

Consider the following Lyapunov function candidate for the closed-loop system composed of (43b) and (44):

$$V_i(\Omega_{\epsilon i}, \tilde{\Theta}_i, \tilde{D}_i) = \frac{1}{2}\Omega_{\epsilon i}^T J_i\Omega_{\epsilon i} + \frac{1}{2}\tilde{\Theta}_i^T\Lambda_i^{-1}\tilde{\Theta}_i + \frac{1}{2}\tilde{D}_i^2, \quad (45)$$

where $\tilde{\Theta}_i = \hat{\Theta}_i - \Theta_i$ and $\tilde{D}_i = \hat{D}_i - D_i$. Then, the derivative of (45) along the solution of system (43b) satisfies

$$\begin{aligned}\dot{V}_i &= \Omega_{\epsilon i}^T J_i\dot{\Omega}_{\epsilon i} + \tilde{\Theta}_i^T\Lambda_i^{-1}\dot{\hat{\Theta}}_i + \tilde{D}_i\dot{\hat{D}}_i \\ &= \Omega_{\epsilon i}^T((-\omega_i^{\times}\mathbf{L}(\omega_i) + \mathbf{L}(\chi_i))\Theta_i + u_i + d_i - M_{\Omega\epsilon i}) \\ &\quad + \tilde{\Theta}_i^T\Lambda_i^{-1}\Lambda_i(-\omega_i^{\times}\mathbf{L}(\omega_i) + \mathbf{L}(\chi_i))^T\Omega_{\epsilon i} \\ &\quad + \tilde{D}_i\mathrm{sign}^T(\Omega_{\epsilon i})\Omega_{\epsilon i}\end{aligned}$$

$$
\begin{aligned}
&= \Omega_{\epsilon i}^T(-\omega_i^\times \mathbf{L}(\omega_i)\Theta_i + \mathbf{L}(\chi_i)\Theta_i - k_{2i}\Omega_{\epsilon i} \\
&\quad + \omega_i^\times \mathbf{L}(\omega_i)\hat{\Theta}_i - \mathbf{L}(\chi_i)\hat{\Theta}_i - \hat{D}_i \mathrm{sign}(\Omega_{\epsilon i}) + d_i - M_{\Omega\epsilon i}) \\
&\quad + \tilde{\Theta}_i^T(-\omega_i^\times \mathbf{L}(\omega_i) + \mathbf{L}(\chi_i))^T\Omega_{\epsilon i} + \tilde{D}_i \mathrm{sign}^T(\Omega_{\epsilon i})\Omega_{\epsilon i} \\
&= -k_{2i}\|\Omega_{\epsilon i}\|^2 - \|\Omega_{\epsilon i}\|\hat{D}_i + \Omega_{\epsilon i}^T d_i - \Omega_{\epsilon i}^T M_{\Omega\epsilon i} + \|\Omega_{\epsilon i}\|\tilde{D}_i \\
&\le -k_{2i}\|\Omega_{\epsilon i}\|^2 - \|\Omega_{\epsilon i}\|\hat{D}_i + \|\Omega_{\epsilon i}\|D_i - \Omega_{\epsilon i}^T M_{\Omega\epsilon i} + \|\Omega_{\epsilon i}\|\tilde{D}_i \\
&= -k_{2i}\Omega_{\epsilon i}^T\Omega_{\epsilon i} - \Omega_{\epsilon i}^T M_{\Omega\epsilon i} \\
&\le -k_{2i}\|\Omega_{\epsilon i}\|^2 + \frac{k_{2i}}{2}\|\Omega_{\epsilon i}\|^2 + \frac{1}{2k_{2i}}\|M_{\Omega\epsilon i}\|^2 \\
&= -\frac{k_{2i}}{2}\|\Omega_{\epsilon i}\|^2 + \frac{1}{2k_{2i}}\|M_{\Omega\epsilon i}\|^2. \qquad (46)
\end{aligned}
$$

By Remark 3.4, $M_{\Omega\epsilon i}(t)$ decays to zero exponentially. Thus, there exists a positive constant α such that, for all $t \ge 0$, $\int_0^t \frac{1}{2k_{2i}}\|M_{\Omega\epsilon i}(\tau)\|^2 d\tau < \alpha$. Thus, for all $t \ge 0$,

$$
\begin{aligned}
V_i(t) &= V_i(0) + \int_0^t \dot{V}_i(\tau)d\tau, \\
&\le V_i(0) + \int_0^t \frac{1}{2k_{2i}}\|M_{\Omega\epsilon i}(\tau)\|^2 d\tau - \int_0^t \frac{k_{2i}}{2}\|\Omega_{\epsilon i}(\tau)\|^2 d\tau, \\
&\le V_i(0) + \alpha - \int_0^t \frac{k_{2i}}{2}\|\Omega_{\epsilon i}(\tau)\|^2 d\tau, \\
&\le V_i(0) + \alpha. \qquad (47)
\end{aligned}
$$

Therefore, $V_i(\Omega_{\epsilon i}, \tilde{\Theta}_i, \tilde{D}_i)$ is bounded for all $t \ge 0$. As a result, $\Omega_{\epsilon i}$, $\hat{\Theta}_i$, and $\hat{D}_i$ are all bounded.

Let

$$\Psi_i(t) = \int_0^t \frac{k_{2i}}{2}\Omega_{\epsilon i}(\tau)^T J_i \Omega_{\epsilon i}(\tau)d\tau \qquad (48)$$

and $\lambda_{\max}(J_i)$ be the maximal eigenvalue of the inertia matrix J_i. Then, for all $t \ge 0$, from the last two inequalities of (47), we have

$$
\begin{aligned}
\Psi_i(t) &= \int_0^t \frac{k_{2i}}{2}\Omega_{\epsilon i}(\tau)^T J_i \Omega_{\epsilon i}(\tau)d\tau \\
&\le \lambda_{\max}(J_i)\int_0^t \frac{k_{2i}}{2}\|\Omega_{\epsilon i}(\tau)\|^2 d\tau \\
&\le \lambda_{\max}(J_i)(V_i(0) + \alpha - V_i(t)) \\
&\le \lambda_{\max}(J_i)(V_i(0) + \alpha). \qquad (49)
\end{aligned}
$$

Since $\dot{\Psi}_i(t) = \frac{k_{2i}}{2}\Omega_{\epsilon i}(t)^T J_i \Omega_{\epsilon i}(t) \ge 0$, $\lim_{t\to\infty}\Psi_i(t)$ exists.

Moreover, we have

$$
\begin{aligned}
\ddot{\Psi}_i &= k_{2i}\Omega_{\epsilon i}^T J_i \dot{\Omega}_{\epsilon i} \\
&= k_{2i}\Omega_{\epsilon i}^T(-\omega_i^\times \mathbf{L}(\omega_i)\Theta_i + \mathbf{L}(\chi_i)\Theta_i + u_i + d_i - M_{\Omega\epsilon i}) \\
&= k_{2i}\Omega_{\epsilon i}^T(-k_{2i}\Omega_{\epsilon i} + \omega_i^\times \mathbf{L}(\omega_i)\tilde{\Theta}_i - \mathbf{L}(\chi_i)\tilde{\Theta}_i \\
&\quad - \hat{D}_i \mathrm{sign}(\Omega_{\epsilon i}) + d_i - M_{\Omega\epsilon i}) \\
&= -k_{2i}^2\|\Omega_{\epsilon i}\|^2 + k_{2i}\Omega_{\epsilon i}^T\omega_i^\times \mathbf{L}(\omega_i)\tilde{\Theta}_i - k_{2i}\Omega_{\epsilon i}^T\mathbf{L}(\chi_i)\tilde{\Theta}_i \\
&\quad - k_{2i}\hat{D}_i\|\Omega_{\epsilon i}\| + k_{2i}\Omega_{\epsilon i}^T d_i - k_{2i}\Omega_{\epsilon i}^T M_{\Omega\epsilon i}. \qquad (50)
\end{aligned}
$$

Since the communication graph $\mathcal{G}_{\sigma(t)}$ is switching by Assumption 2, Δ_i defined in (37) is piecewise continuous. As a result, from (36) and (40), $M_{\Omega\epsilon i}$ is also piecewise continuous. Since both $M_{\Omega\epsilon i}$ and d_i in (50) are piecewise continuous, $\ddot{\Psi}_i$ may be discontinuous at infinitely many points. Therefore, we cannot invoke the Barbalat's Lemma to conclude $\lim_{t\to\infty}\dot{\Psi}_i(t) = 0$. Nevertheless, we can resort to the generalized Barbalat's Lemma as can be found from [13, Corollary 2.5]. For this purpose, we need to show that there exists a time sequence $\{\tau_j : j = 0, 1, 2, \dots\}$ and dwell time $\tau > 0$ satisfying $\tau_0 = 0$, $\tau_{j+1} - \tau_j \ge \tau > 0$ for all $j = 0, 1, 2, \dots$, such that Ψ_i satisfies the following:

(a) $\lim_{t\to\infty}\Psi_i(t)$ exists and is finite.
(b) $\Psi_i(t)$ is twice differentiable on each time interval $[\tau_j, \tau_{j+1})$.
(c) $\ddot{\Psi}_i(t)$ is uniformly bounded over $[0, \infty)$.

We have verified condition (a). Since the discontinuities are caused by the discontinuities of the switching graph $\mathcal{G}_{\sigma(t)}$ and the discontinuities of the external disturbances d_i, by Assumption 3, $\Psi_i(t)$ also satisfies condition (b). We only need to show that $\ddot{\Psi}_i(t)$ is uniformly bounded over $[0, \infty)$. For this purpose, we need to show χ_i is bounded. To show this, we will first prove that $q_{\epsilon i}, \omega_{\epsilon i}, \Pi_i, \Pi_i^d$, and ω_i are all bounded. From (28c), q_0 is a unit quaternion, and $\|q_{\epsilon i}(t)\| = \|\Phi_i^{\,*}\|$ by Remark 3.3, we obtain $q_{\epsilon i}$ and hence $\hat{q}_{\epsilon i}$ are bounded. Then, with bounded $\hat{q}_{\epsilon i}$, we can obtain $\omega_{\epsilon i}$ are bounded from (38). In addition, ζ_k is bounded for $k = 1, 2, 3$ by neutrally stable $S_k(\theta_{0k})$, from which we know that both ω_0 and $\dot{\omega}_0$ are bounded because $\omega_{0k} = \mathbf{c}_k\zeta_k$ and $\dot{\omega}_{0k} = \mathbf{c}_k S_k(\theta_{0k})\zeta_k$ from (14c) and (14d). With bounded ω_0 and $\dot{\omega}_0$, from (28b) and (33a), both Π_i and Π_i^d are bounded. Further, with bounded $q_{\epsilon i}$, $\omega_{\epsilon i}$, and Π_i, from (34a) we can obtain ω_i are bounded. Thus, $q_{\epsilon i}, \omega_{\epsilon i}, \Pi_i, \Pi_i^d$, and ω_i are all bounded. As a result, from (42), χ_i are bounded. Now, since $\Omega_{\epsilon i}, \omega_i, \chi_i, \tilde{\Theta}_i, \hat{D}_i$ are proved to be bounded and d_i and $M_{\Omega\epsilon i}$ are bounded by Assumption 1 and Remark 3.4 over $[\tau_j, \tau_{j+1})$, we can further obtain $\ddot{\Psi}_i$ is uniformly bounded over $[0, \infty)$ in the sense that there exists a positive constant K such that

$$\sup_{\tau_j \le t \le \tau_{j+1}, j=0,1,2,\dots} |\ddot{\Psi}_i(t)| \le K, \quad i = 1, \dots, N. \qquad (51)$$

Thus, we can invoke the Generalized Barbalat's Lemma ([13, Corollary 2.5]), which concludes $\dot{\Psi}_i(t) \to 0$ as $t \to \infty$ and hence $\lim_{t\to\infty}\Omega_{\epsilon i}(t) = \mathbf{0}$. Then from Lemma 3.5, we have $\lim_{t\to\infty}\hat{q}_{\epsilon i}(t) = \mathbf{0}$ and $\lim_{t\to\infty}\hat{\omega}_{\epsilon i}(t) = \mathbf{0}$. Finally, (7) follows from Lemma 3.4. □

4. A Numerical Example

In this section, we evaluate our approach by a group of four rigid body systems. The unknown nominal inertial matrices are as follows:

$$J_1 = J_2 = \begin{bmatrix} 0.729 & & \\ & 0.54675 & \\ & & 0.625 \end{bmatrix} \mathrm{kg}\cdot\mathrm{m}^2,$$
$$J_3 = J_4 = \begin{bmatrix} 0.3645 & & \\ & 0.2734 & \\ & & 0.3125 \end{bmatrix} \mathrm{kg}\cdot\mathrm{m}^2. \tag{52}$$

The unknown external disturbances d_i are given as follows:

$$d_i(t) = \begin{bmatrix} D_{1i}\sin(t) \\ D_{2i}\cos(t) \\ D_{3i}\mathrm{square}(t) \end{bmatrix}, \quad i = 1,2,3,4, \tag{53}$$

where D_{1i}, D_{2i}, D_{3i} are unknown positive constants, and

$$\mathrm{square}(t) = \begin{cases} 1, & t \in [2k\pi, (2k+1)\pi), \\ -1, & t \in [(2k+1)\pi, 2(k+1)\pi) \end{cases}, \quad k \in \mathbb{Z}. \tag{54}$$

It can be verified that d_i, $i = 1,2,3,4$, satisfy Assumption 1, and in the simulation we set $D_{1i} = D_{2i} = D_{3i} = 1$.

The desired angular velocity is as follows:

$$\omega_0(t) = \begin{bmatrix} \Omega_{11}\sin(\psi_{011}t + \sigma_{11}) \\ \Omega_{21}\sin(\psi_{021}t + \sigma_{21}) \\ \Omega_{31}\sin(\psi_{031}t + \sigma_{31}) \end{bmatrix} \mathrm{rad/s}. \tag{55}$$

Thus, for $k = 1,2,3$, $n_k = 1$, $\theta_{0k1} = \psi_{0k1}^2$, and $S_k(\theta_{0k})$ in (14c) have the following form:

$$S_1(\theta_{01}) = \begin{bmatrix} 0 & 1 \\ -\theta_{011} & 0 \end{bmatrix}, \quad S_2(\theta_{02}) = \begin{bmatrix} 0 & 1 \\ -\theta_{021} & 0 \end{bmatrix},$$
$$S_3(\theta_{03}) = \begin{bmatrix} 0 & 1 \\ -\theta_{031} & 0 \end{bmatrix}. \tag{56}$$

The switching signal is as follows:

$$\sigma(t) = \begin{cases} 1 & \text{if } 0.4s \le t < 0.4s + 0.1, \\ 2 & \text{if } 0.4s + 0.1 \le t < 0.4s + 0.2, \\ 3 & \text{if } 0.4s + 0.2 \le t < 0.4s + 0.3, \\ 4 & \text{if } 0.4s + 0.3 \le t < 0.4s + 0.4, \end{cases} \tag{57}$$

where $s = 0,1,2,\ldots$. The communication graph given by the switching signal is shown in Fig. 1.

It can be verified that Assumption 2 is satisfied.

Thus, by Theorem 3.2, Problem 1 can be solved by the control law composed of (44), (17), (23), and (25).

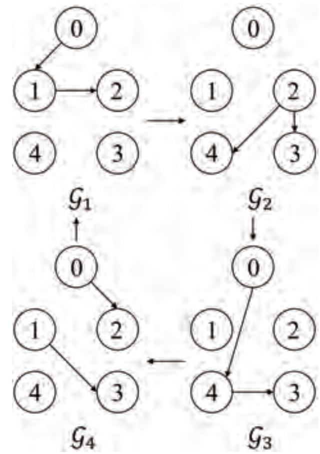

Fig. 1. The communication graph $\mathcal{G}_{\sigma(t)}$ determined by the switching signal (57).

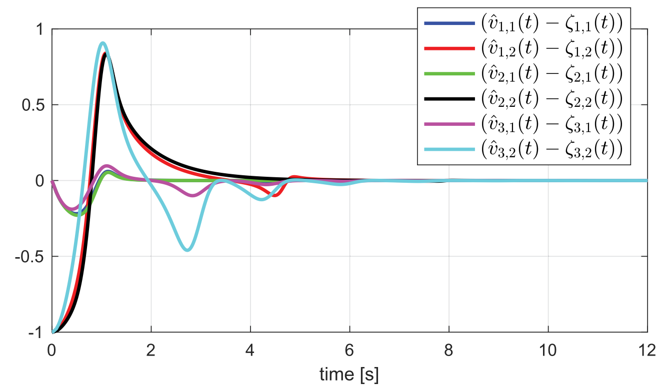

Fig. 2. The response of $(\hat{v}_{k,j} - \zeta_{k,j})$ under exponentially convergent estimator (17).

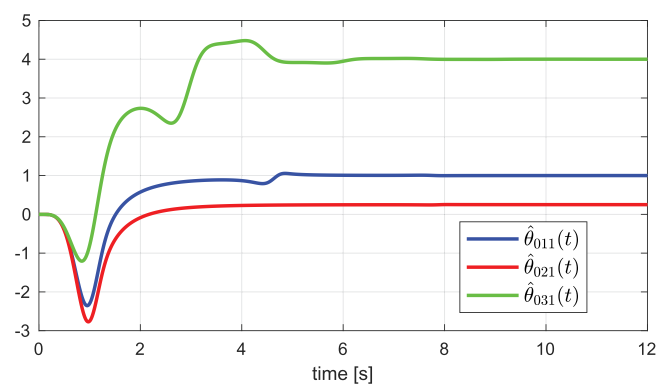

Fig. 3. The response of $\hat{\theta}_{0k1}$ under exponentially convergent estimator (17).

For $k = 1,2,3$, in (17), $\mathbf{d}_k$, are set to col(3, 2) to make the system matrix $\mathbf{D}_k$ Hurwitz, and the positive constant γ_k is set to 1000. For $i = 1,\ldots,N$ and $k = 1,2,3$, the control gains in (25) and (23) are selected to be $\mu_\Phi = 1$, $\mu_{Fk} = 5$,

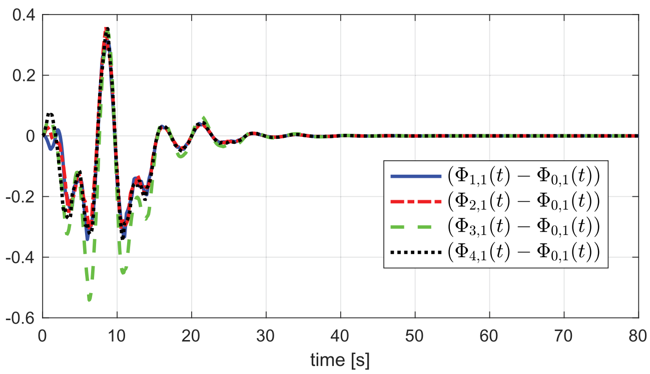

Fig. 4. The response of $(\Phi_{i1}(t) - \Phi_{01}(t))$ of the adaptive distributed observer (25).

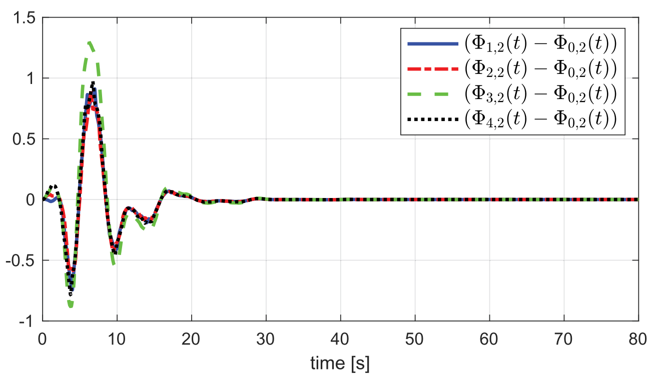

Fig. 5. The response of $(\Phi_{i2}(t) - \Phi_{02}(t))$ of the adaptive distributed observer (25).

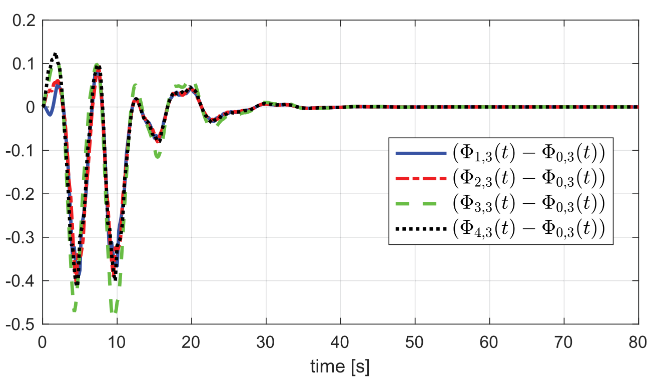

Fig. 6. The response of $(\Phi_{i3}(t) - \Phi_{03}(t))$ of the adaptive distributed observer (25).

$\mu_{1k} = 0.8, \mu_{2k} = 3$. In (42) and (44), we set $k_{1i} = 0.3$ and $k_{2i} = 0.1$, and Λ_i are set as $\Lambda_i = I_6$.

For the purpose of simulation, for $k = 1, 2, 3$, let $\Omega_{k1} = 1$ and $\sigma_{k1} = 0$, and let $\psi_{011} = 1$, $\psi_{021} = 0.5$ and $\psi_{031} = 2$. Also, for $i = 1, 2, 3, 4$ and $k = 1, 2, 3$, the initial conditions are randomly chosen as follows:

$$\hat{\Theta}_1(0) = \mathrm{col}(0.6, 0.5, 0.6, 0, 0, 0),$$

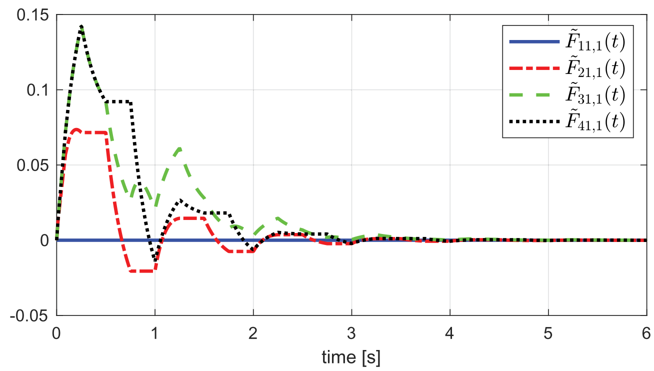

Fig. 7. The response of $\tilde{F}_{i1,1}$ of the adaptive distributed observer (25).

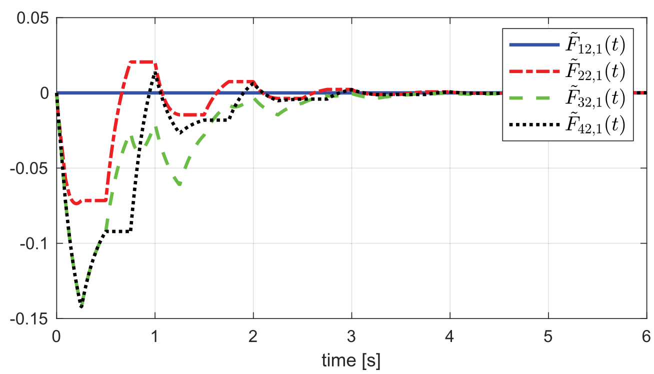

Fig. 8. The response of $\tilde{F}_{i2,1}$ of the adaptive distributed observer (25).

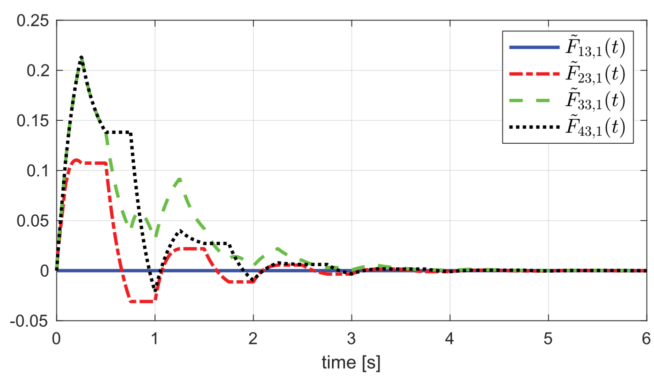

Fig. 9. The response of $\tilde{F}_{i3,1}$ of the adaptive distributed observer (25).

$$\hat{\Theta}_2(0) = \mathrm{col}(0.6, 0.5, 0.6, 0, 0, 0),$$
$$\hat{\Theta}_3(0) = \mathrm{col}(0.3, 0.2, 0.3, 0, 0, 0),$$
$$\hat{\Theta}_4(0) = \mathrm{col}(0.3, 0.2, 0.3, 0, 0, 0),$$
$$\hat{D}_1(0) = 0.8, \quad \hat{D}_2(0) = 0.8,$$
$$\hat{D}_3(0) = 0.8, \quad \hat{D}_4(0) = 0.8,$$
$$q_0(0) = \mathrm{col}(0, 0, 0, 1),$$

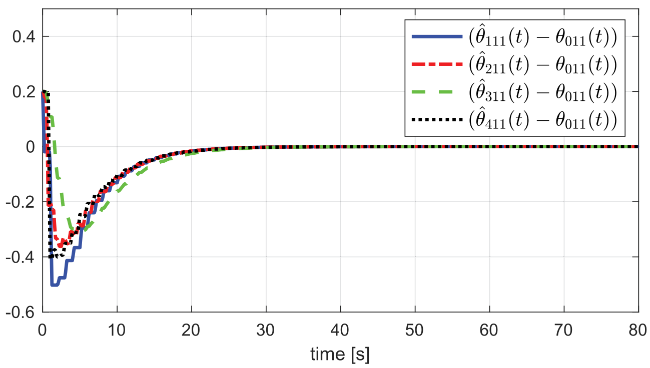

Fig. 10. The response of $(\hat{\theta}_{i11}(t)-\theta_{011}(t))$ of the adaptive distributed observer (23).

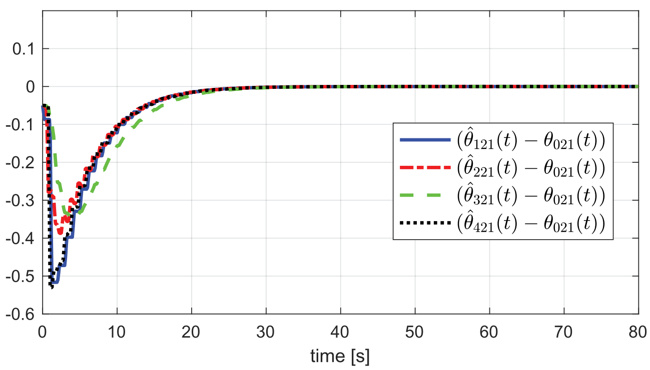

Fig. 11. The response of $(\hat{\theta}_{i21}(t)-\theta_{021}(t))$ of the adaptive distributed observer (23).

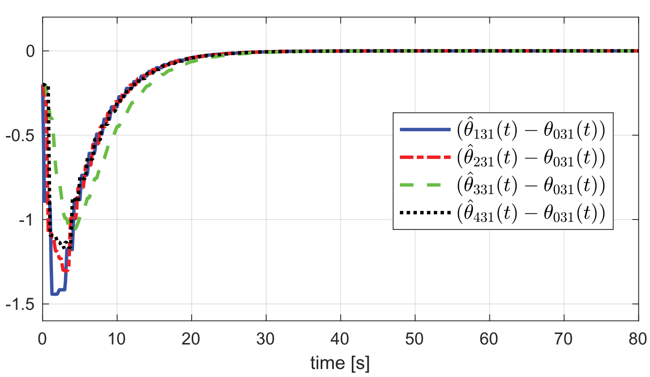

Fig. 12. The response of $(\hat{\theta}_{i31}(t)-\theta_{031}(t))$ of the adaptive distributed observer (23).

$$q_1(0)=\mathrm{col}\left(\sin\frac{\pi}{8},0,0,\cos\frac{\pi}{8}\right),$$
$$q_2(0)=\mathrm{col}\left(0,\sin\frac{\pi}{6},0,\cos\frac{\pi}{6}\right),$$
$$q_3(0)=\mathrm{col}\left(0,0,\sin\frac{\pi}{4},\cos\frac{\pi}{4}\right),$$
$$q_4(0)=\mathrm{col}(1,0,0,0),$$
$$\omega_i(0)=\mathrm{col}(0.8,0.8,0.8),\quad \eta_k(0)=\mathrm{col}(0,0),$$

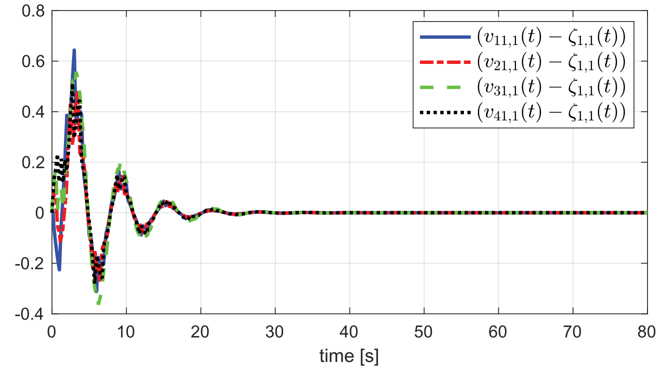

Fig. 13. The response of $(v_{i1,1}(t)-\zeta_{1,1}(t))$ of the adaptive distributed observer (23).

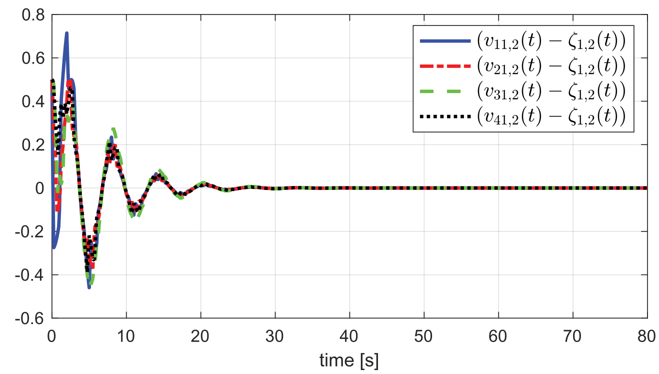

Fig. 14. The response of $(v_{i1,2}(t)-\zeta_{1,2}(t))$ of the adaptive distributed observer (23).

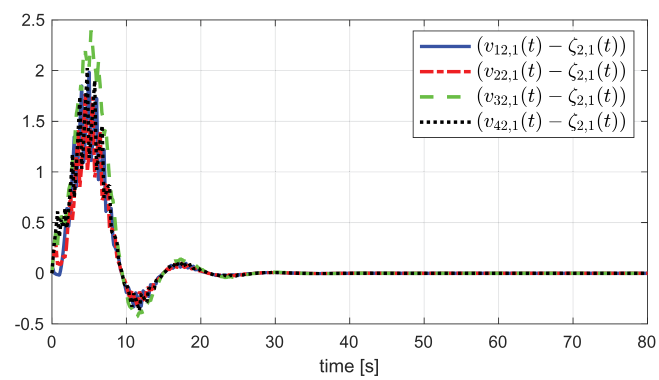

Fig. 15. The response of $(v_{i2,1}(t)-\zeta_{2,1}(t))$ of the adaptive distributed observer (23).

$$\begin{aligned}
\xi_k(0)&=\mathrm{col}(0,0),\quad \hat{\theta}_{0k1}(0)=0.5,\\
\Phi_i(0)&=\mathrm{col}(0,0,0,1),\quad \hat{\theta}_{i11}(0)=1.2,\\
\hat{\theta}_{i21}(0)&=0.2,\quad \hat{\theta}_{i31}(0)=3.8,\\
v_i(0)&=\mathrm{col}(0,1.5,0,2,0,1.5),\\
F_{i1}(0)&=[1.2\quad 0],\quad F_{i2}(0)=[0.8\quad 0],\quad F_{i3}(0)=[1.3\quad 0].
\end{aligned}\tag{58}$$

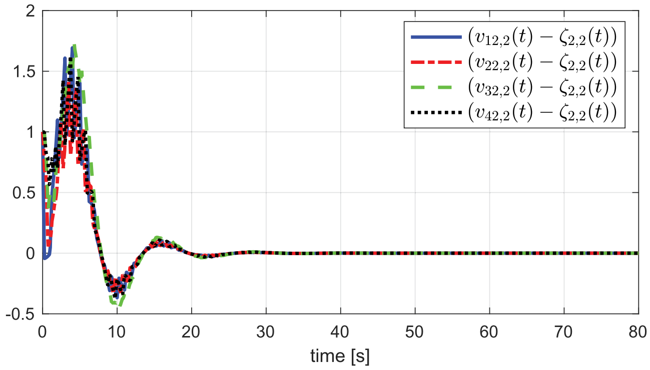

Fig. 16. The response of $(v_{i2,2}(t) - \zeta_{2,2}(t))$ of the adaptive distributed observer (23).

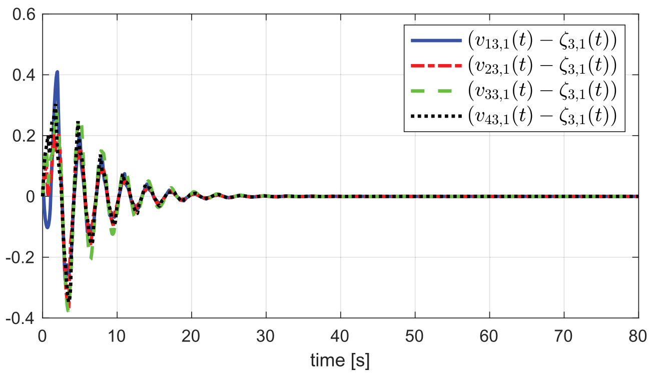

Fig. 17. The response of $(v_{i3,1}(t) - \zeta_{3,1}(t))$ of the adaptive distributed observer (23).

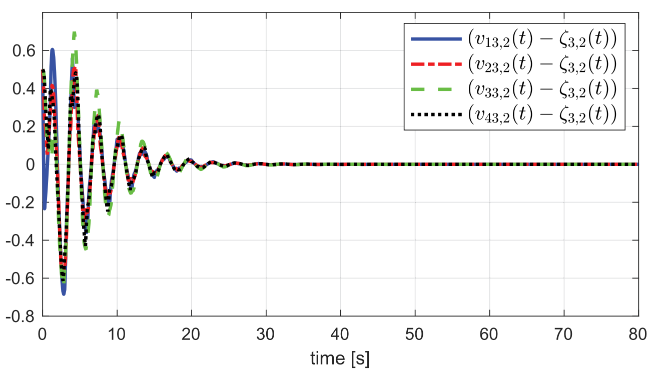

Fig. 18. The response of $(v_{i3,2}(t) - \zeta_{3,2}(t))$ of the adaptive distributed observer (23).

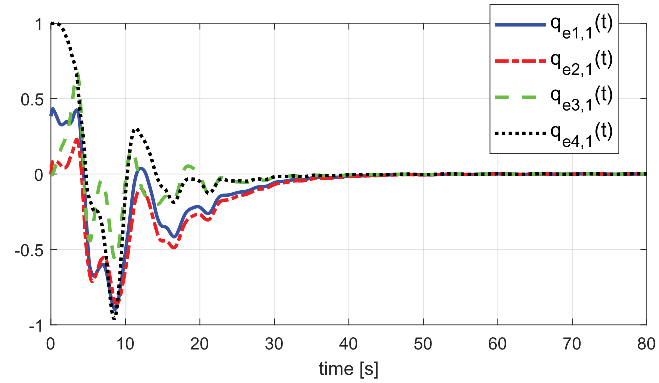

Fig. 19. The response of $q_{ei,1}(t)$ under the control law (44).

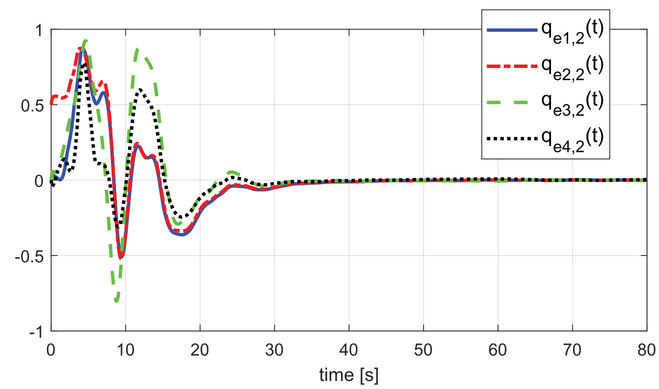

Fig. 20. The response of $q_{ei,2}(t)$ under the control law (44).

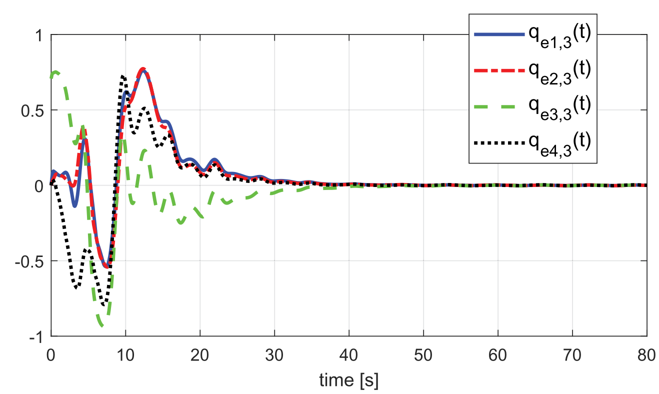

Fig. 21. The response of $q_{ei,3}(t)$ under the control law (44).

Let us first evaluate the performance of the exponentially convergent estimator (17). Let $\hat{v}_{k,j}$ and $\zeta_{k,j}$ represent the jth element of $\hat{v}_k$ and ζ_k, respectively for $k = 1, 2, 3$, where $\hat{v}_k(t)$ is given in (17d). Then the time responses of $(\hat{v}_{k,j} - \zeta_{k,j})$ and $\hat{\theta}_{0k1}$ for $k = 1, 2, 3$ are shown in Figs. 2 and 3, respectively.

From Fig. 2, we can see that all the estimated errors of ζ_k, $k = 1, 2, 3$, converge to zero asymptotically. From Fig. 3, we can see that $\lim_{t\to\infty} \hat{\theta}_{011} = 1$, $\lim_{t\to\infty} \hat{\theta}_{021} = 0.25$, and $\lim_{t\to\infty} \hat{\theta}_{031} = 4$. Thus, we can identify that the unknown frequencies are $\psi_{011} = \sqrt{\theta_{011}} = 1$, $\psi_{021} = \sqrt{\theta_{021}} = 0.5$, and $\psi_{031} = \sqrt{\theta_{031}} = 2$.

To evaluate the performance of the adaptive distributed observer (25) and (23), let $\Phi_{i,j}$, $v_{ik,j}$, and $\tilde{F}_{ik,j}$ be the jth element of Φ_i, v_{ik}, and $\tilde{F}_{ik}$, respectively. Figures 4–6 show the

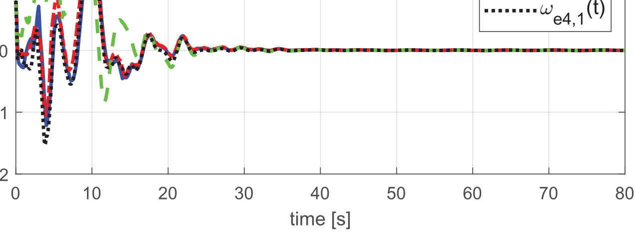

Fig. 22. The response of $\omega_{ei,1}(t)$ under the control law (44).

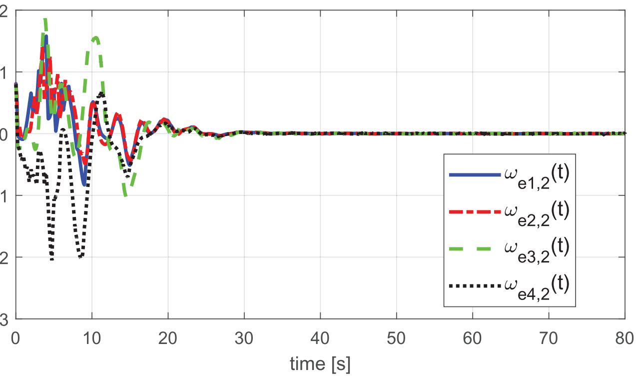

Fig. 23. The response of $\omega_{ei,2}(t)$ under the control law (44).

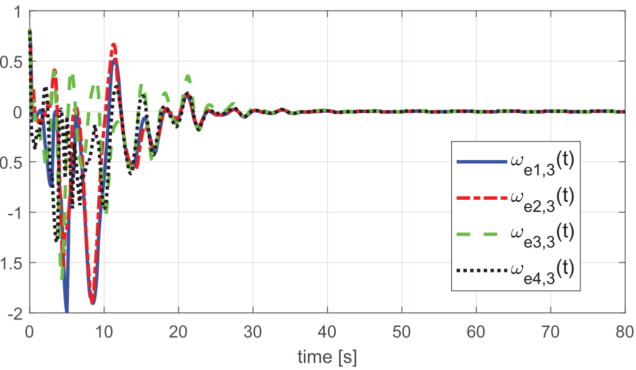

Fig. 24. The response of $\omega_{ei,3}(t)$ under the control law (44).

me response of $(\Phi_i(t) - \Phi_0(t))$, Figs. 7–9 show the time sponse of $\tilde{F}_{ik,1}(t)$. Since $\tilde{F}_{ik,2}(0) = 0$ for all $i = 1, \ldots, N$ and $= 1, 2, 3$, $\tilde{F}_{ik,2}(t) = 0$ for all $t \geq 0$ are not shown here. gures 10–12 show the time response of $(\hat{\theta}_{ik1}(t) - \theta_{0k1}(t))$ r $k = 1, 2, 3$, and Figs. 13–18 show the time response of $_{ik}(t) - \zeta_k(t))$ for $k = 1, 2, 3$. As expected, the adaptive dis-

system converge to zero asymptotically.

5. Conclusions

This paper has studied the leader-following attitude conse sus with disturbance rejection problem of multiple uncerta rigid body systems with an uncertain leader over jointly co nected switching networks. To address this problem, we ha developed an adaptive distributed observer-based feedba control law for uncertain leader systems. Our proposed co trol law only relies on the output of the leader system for th angular velocities and is able to reject any bounded extern disturbances with unknown bounds.

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