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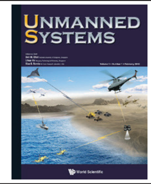

Event-Triggered Feedback Optimization for Uncertain Nonlinear Systems: A Small-Gain Approach

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This paper addresses the event-triggered feedback optimization problem for a class of nonlinear systems in strict-feedback form with dynamic uncertainties. The system has access to real-time gradients of the objective function along its output trajectory, which is a milder requirement than having access to the objective function's analytical form. By employing backstepping and gain assignment techniques, two event-triggered mechanisms and a feedback optimization protocol are developed that construct the closed-loop system as a feedback interconnection of multiple Input-to-State Stable (ISS) subsystems. The small-gain theorem is applied to guarantee closed-loop stability, ensuring that the proposed protocol drives the system output to the optimal solution while excluding Zeno behavior. We further relax the uncertain system dynamics to allow nonvanishing terms at the equilibrium point and bounds given by nonlinear functions involving unknown parameters. Unlike existing small-gain methods for event-triggered control, the proposed approach avoids constructing set-valued maps and incorporates the optimization objective into the small-gain analysis. The effectiveness of the approach is demonstrated through its application to a source-seeking problem.

Keywords: Feedback optimization; event-triggered control; uncertain nonlinear systems; small-gain theorem; input-to-state stability.

1. Introduction

Uncertain nonlinear systems are widely used to model real-world dynamics such as vehicles, robots, electrical circuits, and oscillators. Addressing the challenges posed by nonlinearities and uncertainties is central to analyzing and controlling such systems [1–3], particularly when the system involves multiple subsystems [4–9] and high-order uncertain dynamics [10, 11]. The nonlinear small-gain theorem [12, 13] provides a powerful method to handle these difficulties. Its key insight is that a feedback interconnection of two Input-to-State Stable (ISS) subsystems [14] remains ISS if the loop gain is smaller than the identity function. Compared with other analytical tools, it avoids constructing Lyapunov functions for the overall closed-loop system and can address both internal Lyapunov stability and external input-output stability. Over the past decades, the theorem has been extended to Lyapunov-function formulations [15], systems with saturation constraints [16], discrete-time [17] and hybrid dynamics [18], and finite-time convergence settings [19, 20]. It has also been successfully applied to controller design for stabilization of nonholonomic systems [21], consensus of multi-agent systems [22], safety assurance of mobile agents [23], and prescribed-time regulation of disturbed nonlinear systems [24].

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A recent research trend in uncertain nonlinear systems is feedback optimization, which aims to steer a plant toward the optimal solution of an optimization problem using real-time gradient information. This approach eliminates the need for offline computation of the optimizer [25–27]. Since the optimization protocol and the physical system are inherently interconnected, neglecting this coupling may lead to closed-loop instability [28]. To ensure the stability of the interconnection, related results have been established for single nonlinear systems [26,29,30], as well as for linear [25, 31–33] and nonlinear [27, 34, 35] multi-agent systems.

However, all these approaches require continuous access to both the plant state and the real-time gradient, which can incur significant computational, communication, and actuation costs. To mitigate these demands, event-triggered feedback optimization protocols have been studied for single linear systems [36] and for linear [37] and nonlinear [38] multi-agent systems. In particular, [38] proposed an adaptive dynamic event-triggered distributed protocol for nonlinear agents with parametric uncertainties. To the best of our knowledge, however, event-triggered feedback optimization for nonlinear systems with dynamic uncertainties has not yet been addressed.

This paper proposes a small-gain-based approach to solve the event-triggered feedback optimization problem for uncertain nonlinear systems in strict-feedback form. The objective function is assumed to satisfy a monotonicity condition with a Lipschitz-continuous gradient. By exploiting the system state and real-time gradient information along the output trajectory, two event-triggered mechanisms and a feedback optimization protocol are developed to form the closed-loop system as a feedback interconnection of multiple ISS subsystems. The small-gain theorem is employed to establish sufficient conditions ensuring closed-loop stability and convergence of the system output to the optimal solution, while excluding Zeno behavior, i.e. ensuring that no infinite number of triggering times occurs within any finite time interval. To demonstrate the systematic nature and broader applicability of the proposed small-gain method, we also show how the controller design can be extended to more general classes of systems, including those with nonvanishing nonlinearities and unknown parameters. Furthermore, unlike previous small-gain methods for event-triggered stabilization of nonlinear systems with dynamic [39] and parametric [40] uncertainties or nonholonomic constraints [41, 42], the present approach avoids constructing set-valued maps and explicitly incorporates the optimization objective into the small-gain analysis.

The remainder of this paper is organized as follows. Section 2 presents the necessary preliminaries. Section 3 formulates the event-triggered feedback optimization problem. Section 4 develops the protocol design methodology. Section 5 demonstrates the effectiveness of the proposed approach through a source-seeking example. Section 6 concludes the paper.

2. Preliminaries

2.1. *Notations*

In this paper, $\mathcal{N} = \{1, 2, \dots, n\}$. $\mathbb{R}_+$ and $\mathbb{Z}_+$ denote the set of all non-negative real numbers and integers, respectively. $\mathbb{R}^m$ represents the m-dimensional Euclidean space. Id denotes the identity function. For a vector $x \in \mathbb{R}^m$, $|x|$ denotes its Euclidean norm. For a measurable and locally essentially bounded signal $y : \mathbb{R}_+ \to \mathbb{R}^m$, its infinity norm is defined as $\|y\|_\infty = \sup_{t\geq 0} |y(t)|$. For a vector $x \in \mathbb{R}^m$ and continuously differentiable functions $c : \mathbb{R}^m \to \mathbb{R}$ and $\alpha : \mathbb{R}^m \to \mathbb{R}^m$, $\nabla c(x)$ and $\frac{\partial \alpha(x)}{\partial x}$ denote their gradient and Jacobian matrix at x, respectively. A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}$, denoted as $f \in \mathcal{K}$, if $f(0) = 0$ and f is continuous and strictly increasing. It belongs to class $\mathcal{K}_\infty$ if $f \in \mathcal{K}$ and $\lim_{t\to\infty} f(t) = \infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if $\beta(s, t) \in \mathcal{K}$ for any fixed $t \geq 0$ and $\beta(s, t)$ is decreasing in t with $\lim_{t\to\infty} \beta(s, t) = 0$ for each fixed $s \geq 0$. For functions $g, h : \mathbb{R}_+ \to \mathbb{R}_+$, $g \circ h$ denotes their composition, i.e. $g \circ h(\tau) = g(h(\tau))$ for $\tau \in \mathbb{R}_+$.

2.2. *Small-gain theory*

Consider the nonlinear system

$$\dot{x} = f(x, d), \tag{1}$$

where $x \in \mathbb{R}^q$ is the state, $d \in \mathbb{R}^p$ is the input, and $f : \mathbb{R}^q \times \mathbb{R}^p$ is locally Lipschitz and satisfies $f(0, 0) = 0$.

The system (1) is said to be ISS [14, 43] with x as the state and d as the input, if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K} \cup \{0\}$ such that for any $x(0) \in \mathbb{R}^q$ and any measurable and locally essentially bounded d,

$$|x(t)| \leq \max\{\beta(|x(0)|, t), \gamma(\|d\|_\infty)\}, \tag{2}$$

for all $t \in \mathbb{R}_+$. The function γ is called the ISS gain of (1). For the system, if $\lim_{t\to\infty} d(t) = 0$, then $\lim_{t\to\infty} x(t) = 0$.

We now recall the cyclic small-gain theorem [43, 44], which generalizes the original two-subsystem result in [12] to large-scale systems with multiple subsystems.

Theorem 2.1 (Cyclic-Small-Gain Theorem). *Consider a large-scale system composed of n subsystems whose jth subsystem dynamics are*

$$\dot{x}_j = f_j(x, d_j), \tag{3}$$

where $x = [x_1^T, \dots, x_n^T]^T \in \mathbb{R}^q$ is the state with $x_j \in \mathbb{R}^{q_j}$, $d = [d_1^T, \dots, d_n^T]^T \in \mathbb{R}^p$ is the input with $d_j \in \mathbb{R}^{p_j}$, and $f_j : \mathbb{R}^{q_j} \times \mathbb{R}^{p_j}$ is locally Lipschitz and satisfies $f_j(0, 0) = 0$ for $j \in \mathcal{N}$.

Suppose that each x_j-subsystem is ISS with x_j as the state, x_i's $(i \in \mathcal{N}\backslash\{j\})$ and d_j as the inputs, and $\gamma_j^i \in \mathcal{K} \cup \{0\}$ and $\gamma_j^{d_j} \in \mathcal{K} \cup \{0\}$ as the ISS gains, i.e.

$$|x_j(t)| \leq \max_{i \in \mathcal{N}\backslash\{j\}} \{\beta_j(|x_j(0)|, t), \gamma_j^i(\|x_i\|_\infty), \gamma_j^{d_j}(\|d_j\|_\infty)\}, \tag{4}$$

for some $\beta_j \in \mathcal{KL}$ and all $t \in \mathbb{R}_+$. Then, the large-scale system is ISS with x as the state and d as the input if for every simple loop $j_1 \to j_2 \to \dots \to j_s \to j_1$ ($j_i \neq j_k$ if $i \neq k$) in the interconnection, the following small-gain condition holds

$$\gamma_{j_1}^{j_s} \circ \gamma_{j_s}^{j_{s-1}} \circ \dots \circ \gamma_{j_3}^{j_2} \circ \gamma_{j_2}^{j_1} < \mathrm{Id}. \tag{5}$$

3. Problem Formulation

Consider the uncertain nonlinear system

$$\dot{x}_1 = x_2 + f_1(x_1, d_1), \tag{6a}$$

$$\dot{x}_j = x_{j+1} + f_j(x_1, \dots, x_j, d_j), \quad j = 2, \dots, n-1, \tag{6b}$$

$$\dot{x}_n = u + f_n(x_1, \dots, x_n, d_n), \tag{6c}$$

$$y = x_1, \tag{6d}$$

where $x = [x_1^T, \dots, x_n^T]^T$ is the state with $x_j \in \mathbb{R}^m$ for $j \in \mathcal{N}$, $d = [d_1^T, \dots, d_n^T]^T$ is the unknown time-varying disturbance with $d_j \in \mathbb{R}^{p_j}$ for $j \in \mathcal{N}$, $u \in \mathbb{R}^m$ is the control input, and $y \in \mathbb{R}^m$ is the output. For $j = 1, \dots, n$, $f_j : \mathbb{R}^m \times \dots \times \mathbb{R}^m \times \mathbb{R}^{p_j} \to \mathbb{R}^m$ is an unknown locally Lipschitz function.

A continuously differentiable function $c : \mathbb{R}^m \to \mathbb{R}$ is used to evaluate the output y, and the system performance improves as $c(y)$ decreases. We suppose that the optimization problem

$$\min_y \ c(y), \tag{7}$$

has a unique optimal solution $y_* = \arg\min_y \ c(y)$.

The objective of this paper is to design an *event-triggered feedback optimization protocol* that generates the control input u, such that for any initial state $x(0) \in \mathbb{R}^{nm}$, the system output converges to the optimizer, i.e.

$$\lim_{t \to \infty} y(t) = \arg\min_y \ c(y). \tag{8}$$

The event-triggered feedback optimization protocol consists of two components: (i) two event-triggered mechanisms that determine the time sets $\{\underline{t}_{\underline{k}} : \underline{k} \in \mathbb{Z}_+\}$ and $\{t_k : k \in \mathbb{Z}_+\}$ and (ii) a feedback optimization protocol that uses the sampled data $\nabla c(y(\underline{t}_{\underline{k}}))$ and $x(t_k)$ to achieve (8). Between consecutive events, zero-order holders maintain the sampled values, yielding the piecewise-constant signals $x^s : \mathbb{R}_+ \to \mathbb{R}^{nm}$ and $\nabla c(y^{\underline{s}}) : \mathbb{R}_+ \to \mathbb{R}^m$ defined by $x^s(t) = x(t_k)$ for $t \in [t_k, t_{k+1})$ and $\nabla c(y^{\underline{s}}(t)) = \nabla c(y(\underline{t}_{\underline{k}}))$ for $t \in [\underline{t}_{\underline{k}}, \underline{t}_{\underline{k}+1})$.

Remark 3.1. As discussed in [25–27], assume the availability of the real-time, output-dependent gradient $\nabla c(y)$ is less restrictive than requiring explicit knowledge of the analytical form of the mapping $\nabla c : \mathbb{R}^m \to \mathbb{R}^m$. Additionally, the two triggering mechanisms are motivated by distinct objectives: the gradient-triggered mechanism focuses on optimization accuracy, whereas the state-triggered mechanism prioritizes tracking performance.

The following assumptions are adopted for the system dynamics and the objective function.

Assumption 3.2. The function f_1 is identically zero, i.e. $f_1(x_1, d_1) = 0$ for all $x_1 \in \mathbb{R}^m$ and $d_1 \in \mathbb{R}^{p_1}$. For each $j = 2, \dots, n$, there exist functions $\psi_{jb} \in \mathcal{K}_\infty$ for $b = 2, \dots, j$, which are linearly bounded near the origin and continuously differentiable on $(0, \infty)$, such that

$$|f_j(x_1, \dots, x_j, d_j)| \leq \sum_{b=2}^{j} \psi_{jb}(|x_b|),$$

for all $[x_1^T, \dots, x_j^T]^T \in \mathbb{R}^{jm}$ and $d_j \in \mathbb{R}^{p_j}$.

Assumption 3.3. The function c satisfies the following strong monotonicity condition at the minimum point y_*:

$$(y - y_*)^T(\nabla c(y) - \nabla c(y_*)) \geq \omega|y - y_*|^2, \tag{9}$$

with ω being a positive constant. Additionally, the gradient ∇c is ϑ-Lipschitz with constant $\vartheta > 0$, i.e., for any $y, \hat{y} \in \mathbb{R}^m$,

$$|\nabla c(y) - \nabla c(\hat{y})| \leq \vartheta|y - \hat{y}|. \tag{10}$$

Remark 3.4. Under Assumption 3.2, $f_j(x_1, 0, \dots, 0, d_j) = 0$ holds for all $x_1 \in \mathbb{R}^m$, $d_j \in \mathbb{R}^{p_j}$, and $j = 1, \dots, n$. As illustrated in Sec. 5, many robotic systems satisfy this property [45]. Relaxations of this assumption to more general cases will be discussed in Sec. 4.3. Assumption 3.3 is commonly adopted in feedback optimization studies [25–27,46]. The strong monotonicity condition (9) [38] is more general than strong convexity and, in particular, does not require the function to be convex.

4. Design of Event-Triggered Feedback Optimization Protocols

Following the framework of feedback optimization [25–27], we design a class of feedback optimization protocols in the form of

$$\dot{r} = -\rho \nabla c(y^{\underline{s}}), \tag{11a}$$

$$e_j = x_j - \alpha_{j-1}(e_{j-1}), \quad j = 1, \dots, n, \tag{11b}$$

$$u = \alpha_n(e_n^s), \tag{11c}$$

where $\rho > 0$ is a design parameter, $\alpha_0(e_0) = r$, and for $j = 1, 2, \dots, n$, $\alpha_j : \mathbb{R}^m \to \mathbb{R}^m$ denotes a virtual control law to be determined. The variable $e_n^s \in \mathbb{R}^m$ is defined by

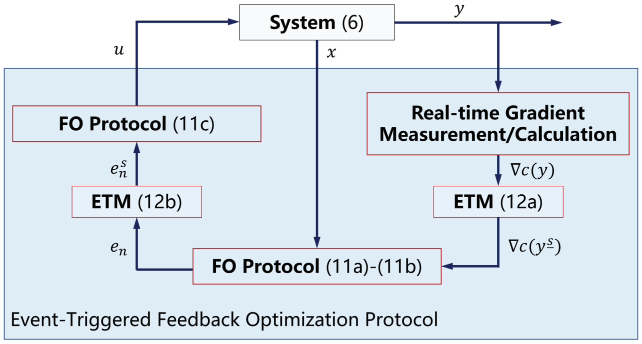

Fig. 1. The diagram of the closed-loop system, including the plant (6), Event-Triggered Mechanisms (ETMs) in (12) and Feedback Optimization (FO) protocol (11).

$e_n^s(t) = e_n(t_k)$ for $t \in [t_k, t_{k+1})$. The triggering instants $\{t_k : k \in \mathbb{Z}_+\}$ and $\{\underline{t}_{\underline{k}} : \underline{k} \in \mathbb{Z}_+\}$ are determined by the following event-triggering mechanisms:

$$\underline{t}_{\underline{k}+1} = \inf\{t > \underline{t}_{\underline{k}} : |\nabla c(y^{\underline{s}}(t)) - \nabla c(y(t))| > \gamma_{\underline{w}}^{c}(|\nabla c(y(t))|)\}, \quad (12a)$$

$$t_{k+1} = \inf\{t > t_k : |e_n^s(t) - e_n(t)| > \gamma_{w_n}^{e_n}(|e_n(t)|)\}, \quad (12b)$$

where $\gamma_{w_n}^{e_n}, \gamma_{\underline{w}}^{c} \in \mathcal{K}_\infty$ are functions to be designed and their inverses $(\gamma_{w_n}^{e_n})^{-1}$ and $(\gamma_{\underline{w}}^{c})^{-1}$ are locally Lipschitz. The diagram of the closed-loop system is depicted in Fig. 1.

Remark 4.1. This design is inspired by [24, 39–41], which developed small-gain-based event-triggered controllers for stabilization problems. Compared with those methods, the design in (11) and (12) introduces two key differences: (i) it does not require prior knowledge of the optimal solution; (ii) the virtual control laws $\alpha_1, \dots, \alpha_{n-1}$, except α_n, are independent of sampled data.

Define the sampling errors

$$\underline{w} = \nabla c(y^{\underline{s}}) - \nabla c(y), \quad (13a)$$

$$w_n = e_n^s - e_n. \quad (13b)$$

Then, (11) can be rewritten as

$$\dot{r} = -\rho(\nabla c(y) + \underline{w}), \quad (14a)$$

$$e_j = x_j - \alpha_{j-1}(e_{j-1}), \quad j = 1, \dots, n, \quad (14b)$$

$$u = \alpha_n(e_n + w_n). \quad (14c)$$

4.1. *Recursive design and ISS properties of the subsystems*

This section establishes a recursive design procedure for the virtual control laws α_j's and derives the ISS properties of all subsystems in the closed-loop system composed of (6), (13) and (14).

Lemma 4.2. *Consider the system consisting of* (6) *and* (14). *If Assumption* 3.2 *holds, then for each* $j = 1, 2, \dots, n$, *any gains* $\gamma_{e_j}^{e_b}, \gamma_{e_j}^{\dot{r}} \in \mathcal{K}_\infty$ $(b = 1, 2, \dots, j+1)$ *whose inverses* $(\gamma_{e_j}^{e_b})^{-1}, (\gamma_{e_j}^{\dot{r}})^{-1}$ *are locally Lipschitz and linearly bounded near the origin, and any constant* $0 < \mu_n < 1$, *there exist a locally Lipschitz map* $\phi_j : \mathbb{R}^m \times \cdots \mathbb{R}^m \times \mathbb{R}^{\bar{p}_j} \to \mathbb{R}^m$ *with* $\bar{p}_j = \sum_{b=1}^{j} p_b$, *functions* $\varphi_j^{e_1}, \dots, \varphi_j^{e_{j+1}}, \varphi_j^{\dot{r}} \in \mathcal{K}_\infty$ *that are locally Lipschitz and linearly bounded near the origin, and a continuously differentiable virtual controller* $\alpha_j : \mathbb{R}^m \to \mathbb{R}^m$ *with* $\alpha_j(0) = 0$, *such that:*

(1) *the time-derivative of* e_j *satisfies:*

$$\dot{e}_j = \alpha_j(e_j + a_j w_n) + e_{j+1} + \phi_j(e_1, \dots, e_j, r, \dot{r}, \bar{d}_j), \quad (15)$$

where $e_{n+1} = 0$, $a_i = 0$ *for* $i = 1, \dots, n-1$, $a_n = 1$, $\bar{d}_j = [d_1^T, \dots, d_j^T]^T$, *and*

$$|\phi_j(e_1, \dots, e_j, r, \dot{r}, \bar{d}_j)| \le \sum_{b=1}^{j} \varphi_j^{e_b}(|e_b|) + \varphi_j^{\dot{r}}(|\dot{r}|), \quad (16)$$

for all $e_1, \dots, e_j, \dot{r} \in \mathbb{R}^m$ *and* $d_j \in \mathbb{R}^{p_j}$;

(2) *the* e_j*-subsystem is ISS, i.e.*

$$|e_j(t)| \le \max_{b=1,2,\dots,j+1}\{e^{-l_j t}|e_j(0)|, \gamma_{e_j}^{e_b}(\|e_b\|_\infty), \gamma_{e_j}^{\dot{r}}(\|\dot{r}\|_\infty), \gamma_{e_j}^{w_n}(\|w_n\|_\infty)\}, \quad (17)$$

for all $t \in \mathbb{R}_+$, *where* l_j *is a positive constant,* $\gamma_{e_n}^{w_n} = \frac{1}{\mu_n}\mathrm{Id}$ *and* $\gamma_{e_i}^{w_n} = 0$ *for* $i = 1, \dots, n-1$.

Proof. We first establish Item (1) by induction and then design the virtual controllers via backstepping and gain assignment to obtain Item (2).

Step 1. The time-derivative of e_1 along the solutions of (6a) and (11b) is

$$\begin{aligned} \dot{e}_1 &= \dot{x}_1 - \dot{r} \\ &= x_2 + f_1(x_1, d_1) - \dot{r} \\ &= \alpha_1(e_1) + e_2 - \dot{r}, \end{aligned} \quad (18)$$

where Assumption 3.2 is used for the last equality. Thus, (15) holds with $\phi_1(e_1, \dot{r}) = -\dot{r}$, and

$$|\phi_1(e_1, \dot{r})| = |\dot{r}|, \quad (19)$$

which means that (16) is satisfied with $\varphi_1^{e_1} = 0.1\mathrm{Id}$ and $\varphi_1^{\dot{r}} = \mathrm{Id}$.

Step i ($1 \le i \le n-2$). Suppose that (15) and (16) hold for $j = i$, i.e.

$$\dot{e}_i = \alpha_i(e_i) + e_{i+1} + \phi_i(e_1, \dots, e_i, r, \dot{r}, \bar{d}_i), \quad (20)$$

$$|\phi_i(e_1,\dots,e_i,r,\dot r,\bar d_i)| \le \sum_{b=1}^{i}\varphi_i^{e_b}(|e_b|)+\varphi_i^{\dot r}(|\dot r|), \tag{21}$$

for some locally Lipschitz functions $\phi_i : \mathbb{R}^m \times \cdots \mathbb{R}^m \times \mathbb{R}^{\bar p_i} \to \mathbb{R}^m$ and $\varphi_i^{e_1},\dots,\varphi_i^{e_i},\varphi_i^{\dot r} \in \mathcal{K}_\infty$.

From (14b) and (20),

$$\begin{aligned}\dot e_{i+1} &= \dot x_{i+1}-\frac{\partial\alpha_i(e_i)}{\partial e_i}\dot e_i\\ &= x_{i+2}+f_{i+1}(x_1,x_2,\dots,x_{i+1},d_{i+1})-\frac{\partial\alpha_i(e_i)}{\partial e_i}\\ &\quad\times(\alpha_i(e_i)+e_{i+1}+\phi_i(e_1,\dots,e_i,r,\dot r,\bar d_i))\\ &= \alpha_{i+1}(e_{i+1})+e_{i+2}+f_{i+1}(x_1,\alpha_1(e_1)\\ &\quad+e_2,\dots,\alpha_i(e_i)+e_{i+1},d_{i+1})-\frac{\partial\alpha_i(e_i)}{\partial e_i}\\ &\quad\times(\alpha_i(e_i)+e_{i+1}+\phi_i(e_1,\dots,e_i,r,\dot r,\bar d_i)),\end{aligned} \tag{22}$$

which matches (15) for $j=i+1$ with

$$\begin{aligned}&\phi_{i+1}(e_1,\dots,e_{i+1},r,\dot r,\bar d_{i+1})\\ &\quad= f_{i+1}(e_1+r,\alpha_1(e_1)+e_2,\dots,\alpha_i(e_i)+e_{i+1},d_{i+1})\\ &\quad\quad-\frac{\partial\alpha_i(e_i)}{\partial e_i}(\alpha_i(e_i)+e_{i+1}+\phi_i(e_1,\dots,e_i,r,\dot r,\bar d)).\end{aligned} \tag{23}$$

From (23), it follows that

$$\begin{aligned}&|\phi_{i+1}(e_1,\dots,e_{i+1},r,\dot r,\bar d_{i+1})|\\ &\quad\le |f_{i+1}(e_1+r,\alpha_1(e_1)+e_2,\dots,\alpha_i(e_i)+e_{i+1},d_{i+1})|\\ &\quad\quad+\left|-\frac{\partial\alpha_i(e_i)}{\partial e_i}(\alpha_i(e_i)+e_{i+1}+\phi_i(e_1,\dots,e_i,r,\dot r,\bar d_i))\right|.\end{aligned} \tag{24}$$

Using Assumption 3.2 and the monotonicity of $\psi_{i+1,b}$ for all $b=2,\dots,i+1$, we have

$$\begin{aligned}&|f_{i+1}(e_1+r,\alpha_1(e_1)+e_2,\dots,\alpha_i(e_i)+e_{i+1},d_{i+1})|\\ &\quad\le\sum_{b=2}^{i+1}\psi_{i+1,b}(|\alpha_{b-1}(e_{b-1})+e_b|)\\ &\quad\le\sum_{b=2}^{i+1}\psi_{i+1,b}(|\alpha_{b-1}(e_{b-1})|+|e_b|)\\ &\quad\le\sum_{b=2}^{i+1}(\psi_{i+1,b}(2|\alpha_{b-1}(e_{b-1})|)+\psi_{i+1,b}(2|e_b|)).\end{aligned} \tag{25}$$

Since α_i is continuous and satisfies $\alpha_i(0)=0$, there exists a locally Lipschitz function $\breve\alpha_i \in \mathcal{K}_\infty$ such that

$$|\alpha_i(e_i)| \le \breve\alpha_i(|e_i|). \tag{26}$$

Hence,

$$\begin{aligned}&|f_{i+1}(e_1+r,\alpha_1(e_1)+e_2,\dots,\alpha_i(e_i)+e_{i+1},d_{i+1})|\\ &\quad\le\sum_{b=2}^{i+1}(\psi_{i+1,b}(2\breve\alpha_{b-1}(|e_{b-1}|))+\psi_{i+1,b}(2|e_b|)).\end{aligned} \tag{27}$$

Moreover, by [21, Lemmas 1 and 2], the properties of α and ϕ in (26) and (21) imply the existence of locally Lipschitz functions $\breve\varphi_{i+1}^{e_1}$, …, $\breve\varphi_{i+1}^{e_{i+1}}$, $\breve\varphi_{i+1}^{\dot r} \in \mathcal{K}_\infty$ such that

$$\begin{aligned}&\left|-\frac{\partial\alpha_i(e_i)}{\partial e_i}(\alpha_i(e_i)+e_{i+1}+\phi_i(e_1,\dots,e_i,r,\dot r,\bar d_i))\right|\\ &\quad\le\sum_{b=1}^{i+1}\breve\varphi_{i+1}^{e_b}(|e_b|)+\breve\varphi_{i+1}^{\dot r}(|\dot r|).\end{aligned} \tag{28}$$

Substituting (27) and (28) into (24) yields

$$\begin{aligned}&|\phi_{i+1}(e_1,\dots,e_{i+1},r,\dot r,\bar d_{i+1})|\\ &\quad\le|f_{i+1}(e_1+r,e_2+\alpha_1(e_1),\dots,e_{i+1}+\alpha_i(e_i),d_{i+1})|\\ &\quad\quad+\left|-\frac{\partial\alpha_i(e_i)}{\partial e_i}(\alpha_i(e_i)+e_{i+1}+\phi_i(e_1,\dots,e_i,r,\dot r,\bar d_i))\right|\\ &\quad\le\sum_{b=2}^{i+1}\psi_{i+1,b}(2\breve\alpha_{b-1}(|e_{b-1}|))+\sum_{b=2,\dots,i+1}\psi_{i+1,b}(2|e_b|)\\ &\quad\quad+\sum_{b=1}^{i+1}\breve\varphi_{i+1}^{e_b}(|e_b|)+\breve\varphi_{i+1}^{\dot r}(|\dot r|),\end{aligned} \tag{29}$$

which verifies (16) for $j=i+1$ and some locally Lipschitz functions $\varphi_{i+1}^{e_1}$, …, $\varphi_{i+1}^{e_{i+1}}$, $\varphi_{i+1}^{\dot r} \in \mathcal{K}_\infty$.

Step n: Noting the definition of u in (14c), the time-derivative of e_n is

$$\begin{aligned}\dot e_n &= \dot x_n-\frac{\partial\alpha_{n-1}(e_{n-1})}{\partial e_{n-1}}\dot e_{n-1}\\ &= u+f_n(x_1,\dots,x_n,d_n)-\frac{\partial\alpha_{n-1}(e_{n-1})}{\partial e_{n-1}}(\alpha_{n-1}(e_{n-1})\\ &\quad+e_n+\phi_{n-1}(e_1,\dots,e_{n-1},r,\dot r,\bar d_{n-1}))\\ &= \alpha_n(e_n+w_n)+f_n(e_1+r,\alpha_1(e_1)+e_2,\dots,\alpha_{n-1}\\ &\quad\times(e_{n-1})+e_n,d_n)-\frac{\partial\alpha_{n-1}(e_{n-1})}{\partial e_{n-1}}(\alpha_{n-1}(e_{n-1})+e_n\\ &\quad+\phi_{n-1}(e_1,\dots,e_{n-1},r,\dot r,\bar d_{n-1})),\end{aligned} \tag{30}$$

completing Item (1) for all $j=1,\dots,n$.

Next, we prove Item (2). For $j = 1, 2, \dots, n$, define

$$V_j(e_j) = |e_j|^2. \tag{31}$$

The time derivative of $V_j(e_j)$ along the solutions of (15) satisfies

$$\begin{aligned}\dot{V}_j(e_j) &= 2e_j^T \dot{e}_j \\ &= 2e_j^T(\alpha_j(e_j) + e_{j+1}) + 2e_j^T \phi_j(e_1, \dots, e_j, r, \dot{r}, \bar{d}_j).\end{aligned} \tag{32}$$

By [21, Lemma 1], there exists a function $\kappa_j : \mathbb{R}_+ \to \mathbb{R}_+$, which is positive, nondecreasing and continuously differentiable on $(0, \infty)$, such that

$$\begin{aligned}&\kappa_j((1-\mu_j)\tau)(1-\mu_j)\tau \\ &\quad \geq \sum_{b=1,\dots,j+1} \varphi_j^{e_b} \circ (\gamma_{e_j}^{e_b})^{-1}(\tau) + \varphi_j^{\dot{r}} \circ (\gamma_{e_j}^{\dot{r}})^{-1}(\tau) + l_j\tau,\end{aligned} \tag{33}$$

for $\tau \in \mathbb{R}_+$, where $\varphi_j^{e_{j+1}} = \mathrm{Id}$, $\gamma_{e_j}^{e_j} = \mathrm{Id}$, and $\mu_i = 0$ for $i = 1, \dots, n-1$.

Choose α_j as

$$\alpha_j(\xi) = -\kappa_j(|\xi|)\xi, \tag{34}$$

for all $\xi \in \mathbb{R}^m$. When

$$\begin{aligned}V_j(e_j) \geq \max_{b=1,2,\dots,j+1} \{&V_j \circ \gamma_{e_j}^{e_b}(|e_b|), V_j \circ \gamma_{e_j}^{\dot{r}}(|\dot{r}|), \\ &V_j \circ \gamma_{e_j}^{w_n}(|w_n|)\},\end{aligned} \tag{35}$$

by (32)–(34), we have

$$\dot{V}_j(e_j) \leq -2l_j V_j(e_j), \tag{36}$$

which means that

$$\begin{aligned}V_j(e_j(t)) \leq \max_{b=1,2,\dots,j+1} \{&e^{-2l_j t} V_j(e_j(0)), V_j \circ \gamma_{e_j}^{e_b}(\|e_b\|_\infty), \\ &V_j \circ \gamma_{e_j}^{\dot{r}}(\|\dot{r}\|_\infty), V_j \circ \gamma_{e_j}^{w_n}(\|w_n\|_\infty)\},\end{aligned} \tag{37}$$

or, equivalently, (17) is satisfied. Thus Item (2) is satisfied ending the proof. □

Remark 4.3. Unlike the analyses in [39–41], the sampling error affects only the e_n-subsystem; the subsystems $e_1, \dots, e_{n-1}$ are unaffected. Consequently, Lemma 4.2 does not require the construction of set-valued maps.

Lemma 4.4. *Under Assumption* 3.3, *the event-triggered mechanisms in* (12) *have the following bounds for all* $t \geq 0$:

$$|\underline{w}(t)| \leq \max\{\gamma_{\underline{w}}^{e_1}(\|e_1\|_\infty), \gamma_{\underline{w}}^{\tilde{r}}(\|\tilde{r}\|_\infty)\}, \tag{38}$$

$$|w_n(t)| \leq \gamma_{w_n}^{e_n}(\|e_n\|_\infty), \tag{39}$$

where $\gamma_{\underline{w}}^{e_1}, \gamma_{\underline{w}}^{\tilde{r}} \in \mathcal{K}_\infty$ *are given by* $\gamma_{\underline{w}}^{e_1}(\tau) = 2\gamma_{\underline{w}}^{c}(2\vartheta\tau)$ *and* $\gamma_{\underline{w}}^{\tilde{r}}(\tau) = 2\gamma_{\underline{w}}^{c}(2\vartheta\tau)$ *for all* $\tau \in \mathbb{R}_+$.

Proof. From (12b)–(12a), we obtain

$$|\underline{w}(t)| \leq \gamma_{\underline{w}}^{c}(|\nabla c(y(t))|), \tag{40}$$

$$|w_n(t)| \leq \gamma_{w_n}^{e_n}(|e_1(t)|), \tag{41}$$

for all $t \geq 0$. Using $\nabla c(y_*) = 0$, the Lipschitz continuity of ∇c, and the weak triangle inequality, we have

$$\begin{aligned}&\gamma_{\underline{w}}^{c}(|\nabla c(y(t))|) \\ &\quad = \gamma_{\underline{w}}^{c}(|\nabla c(y(t)) - \nabla c(r(t)) + \nabla c(r(t)) - \nabla c(y_*)|) \\ &\quad \leq \gamma_{\underline{w}}^{c}(\vartheta|y(t) - r(t)| + \vartheta|r(t) - y_*|) \\ &\quad \leq \gamma_{\underline{w}}^{c}(2\vartheta|y(t) - r(t)|) + \gamma_{\underline{w}}^{c}(2\vartheta|r(t) - y_*|) \\ &\quad \leq \max\{2\gamma_{\underline{w}}^{c}(2\vartheta|e_1(t)|), 2\gamma_{\underline{w}}^{c}(2\vartheta|\tilde{r}(t)|)\} \\ &\quad = \max\{\gamma_{\underline{w}}^{e_1}(|e_1(t)|), \gamma_{\underline{w}}^{\tilde{r}}(|\tilde{r}(t)|)\},\end{aligned} \tag{42}$$

which together with (40), establishes (38). □

Lemma 4.5. *If c satisfies the strong monotonicity condition* (9), *the system* (14a) *is ISS with* $\tilde{r} = r - y_*$ *as the state and* $(e_1, \underline{w})$ *as the inputs. Moreover, for any* $\tilde{r}(0) \in \mathbb{R}^m$, *its trajectories satisfy*

$$|\tilde{r}(t)| \leq \max\{e^{-\frac{1}{3}\rho\omega t}|\tilde{r}(0)|, \gamma_{\tilde{r}}^{e_1}(\|e_1\|_\infty), \gamma_{\tilde{r}}^{w}(\|\underline{w}\|_\infty)\}, \tag{43}$$

for all $t \geq 0$, *where* $\gamma_{\tilde{r}}^{e_1} = \frac{3\vartheta}{\omega}\mathrm{Id}$ *and* $\gamma_{\tilde{r}}^{w} = \frac{3}{\omega}\mathrm{Id}$.

Proof. Define $V_r(\tilde{r}) = |\tilde{r}|^2$. The time derivative of $V_r(\tilde{r})$ along the solutions of (14a) satisfies

$$\begin{aligned}\dot{V}_r(\tilde{r}) &= 2\tilde{r}^T \dot{r} \\ &= -2\rho\tilde{r}^T(\nabla c(y) + \underline{w}) \\ &= -2\rho\tilde{r}^T(\nabla c(r) + \nabla c(y) - \nabla c(r) + \underline{w}) \\ &= -2\rho\tilde{r}^T\nabla c(r) - 2\rho\tilde{r}^T(\nabla c(y) - \nabla c(r)) \\ &\quad - 2\rho\tilde{r}^T\underline{w}.\end{aligned} \tag{44}$$

Since c satisfies the strong monotonicity condition (9),

$$-2\rho\tilde{r}^T\nabla c(r) \leq -2\rho\omega|\tilde{r}|^2. \tag{45}$$

By the Lipschitz property of ∇c,

$$-2\rho\tilde{r}^T(\nabla c(y) - \nabla c(r)) \leq 2\rho\vartheta|\tilde{r}||y - r|, \tag{46}$$

$$-2\rho\tilde{r}^T\underline{w} \leq 2\rho|\tilde{r}||\underline{w}|. \tag{47}$$

Substituting (45), (46) and (47) into (44) yields

$$\dot{V}_r(\tilde{r}) \leq -2\rho\omega|\tilde{r}|^2 + 2\rho\vartheta|\tilde{r}||y - r| + 2\rho|\tilde{r}||\underline{w}|. \tag{48}$$

When $|\tilde{r}| \geq \max\{\frac{3\vartheta}{\omega}|y-r|, \frac{3}{\omega}|\underline{w}|\}$, we have

$$\dot{V}_r(\tilde{r}) \leq -\frac{2}{3}\rho\omega|\tilde{r}|^2 = -\frac{2}{3}\rho\omega V_r(\tilde{r}), \tag{49}$$

which, together with the definitions of V_r and e_1, implies the satisfaction of (43). □

Lemma 4.6. *For system* (14a), *if* ∇c *is* ϑ*-Lipschitz, then*

$$|\dot{r}(t)| \leq \max\{\gamma_{\dot{r}}^{e_1}(\|e_1\|_\infty), \gamma_{\dot{r}}^{\tilde{r}}(\|\tilde{r}\|_\infty), \gamma_{\dot{r}}^{\underline{w}}(\|\underline{w}\|_\infty), \tag{50}$$

where $\gamma_{\dot{r}}^{e_1} = 3\rho\vartheta\mathrm{Id}$, $\gamma_{\dot{r}}^{\tilde{r}} = 3\rho\vartheta\mathrm{Id}$, *and* $\gamma_{\dot{r}}^{\underline{w}} = 3\rho\mathrm{Id}$.

Proof. From (14a) and (14b), it follows that

$$\begin{aligned} |\dot{r}| &= \rho|\nabla c(y) + \underline{w}| \\ &\leq \rho(|\nabla c(y)| + |\underline{w}|) \\ &\leq \rho(\vartheta|e_1| + \vartheta|\tilde{r}| + |\underline{w}|). \end{aligned} \tag{51}$$

Hence,

$$|\dot{r}(t)| \leq \rho(\vartheta\|e_1\|_\infty + \vartheta\|\tilde{r}\|_\infty + \|\underline{w}\|_\infty), \tag{52}$$

which implies (50). This ends the proof. □

4.2. *Closed-loop system analysis based on the small-gain theorem*

By Lemmas 4.2–4.6, the closed-loop system consisting of (6), (13a), (13b) and (14) can be regarded as an interconnection of ISS subsystems. The interconnection relationships among these subsystems are illustrated in Fig. 2. Analyzing the loop gains associated with all simple cycles in the gain graph yields the following main result.

Theorem 4.7. *Consider the system* (6) *under the feedback optimization protocol* (11) *and the event-triggered mechanisms in* (12), *where* α_j*'s are designed according to*

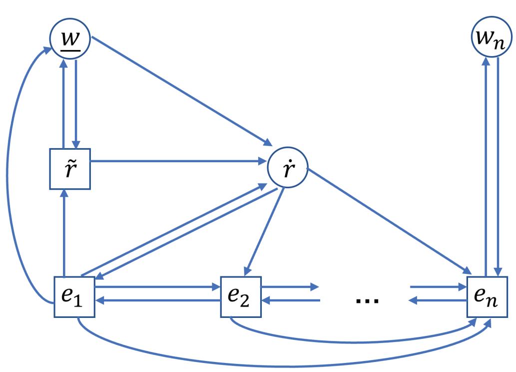

Fig. 2. Interconnection relationships between the ISS subsystems.

Lemma 4.2. *If Assumptions* 3.2 *and* 3.3 *hold, and if* ρ, $\gamma_{e_j}^{e_b}$, $\gamma_{e_j}^{\dot{r}}$, $\gamma_{e_n}^{w_n}$, $\gamma_{w_n}^{e_n}$, *and* $\gamma_{\underline{w}}^{c}$ *satisfy the conditions in Lemma* 4.2 *together with the small-gain conditions*

$$\gamma_{e_\tau}^{e_{\tau+1}} \circ \gamma_{e_{\tau+1}}^{e_{\tau+2}} \circ \cdots \gamma_{e_{\tau+i-1}}^{e_{\tau+i}} \circ \gamma_{e_{\tau+i}}^{e_\tau} < \mathrm{Id}, \tag{53a}$$

$$\gamma_{\dot{r}}^{e_1} \circ \gamma_{e_1}^{e_2} \circ \cdots \gamma_{e_{j-1}}^{e_j} \circ \gamma_{e_j}^{\dot{r}} < \mathrm{Id}, \tag{53b}$$

$$\gamma_{\dot{r}}^{\tilde{r}} \circ \gamma_{\tilde{r}}^{e_1} \circ \gamma_{e_1}^{e_2} \circ \cdots \gamma_{e_{j-1}}^{e_j} \circ \gamma_{e_j}^{\dot{r}} < \mathrm{Id}, \tag{53c}$$

$$\gamma_{\dot{r}}^{\underline{w}} \circ \gamma_{\underline{w}}^{\tilde{r}} \circ \gamma_{\tilde{r}}^{e_1} \circ \gamma_{e_1}^{e_2} \circ \cdots \gamma_{e_{j-1}}^{e_j} \circ \gamma_{e_j}^{\dot{r}} < \mathrm{Id}, \tag{53d}$$

$$\gamma_{\dot{r}}^{\underline{w}} \circ \gamma_{\underline{w}}^{e_1} \circ \gamma_{e_1}^{e_2} \circ \cdots \gamma_{e_{j-1}}^{e_j} \circ \gamma_{e_j}^{\dot{r}} < \mathrm{Id}, \tag{53e}$$

$$\gamma_{\underline{w}}^{\tilde{r}} \circ \gamma_{\tilde{r}}^{\underline{w}} < \mathrm{Id}, \tag{53f}$$

$$\gamma_{w_n}^{e_n} \circ \gamma_{e_n}^{w_n} < \mathrm{Id}, \tag{53g}$$

for $\tau = 1, 2, \ldots, n-1$, $i = 1, 2, \ldots, n-\tau$, *and* $j = 1, 2, \ldots, n$, *then the optimization objective* (8) *is achieved, and the closed-loop system is free of Zeno behavior.*

Proof. By Lemmas 4.4–4.6, the ISS gains $\gamma_{\tilde{r}}^{\underline{w}}$, $\gamma_{\tilde{r}}^{e_1}$, $\gamma_{\dot{r}}^{\underline{w}}$, $\gamma_{\dot{r}}^{\tilde{r}}$, $\gamma_{\dot{r}}^{e_1}$, and $\gamma_{e_n}^{w_n}$ are linear functions, while the remaining ISS gains can be designed as any linear functions. Therefore, the conditions in Lemma 4.2 and the small-gain conditions (53) can be simultaneously satisfied. Applying the cyclic small-gain theorem (Theorem 2.1 in this paper) yields

$$|X(t)| \leq \beta(|X(0)|, t), \tag{54}$$

for some $\beta \in \mathcal{KL}$ and all $t \in \mathbb{R}_+$, where $X = [e_1^T, \ldots, e_n^T, \tilde{r}^T, \dot{r}^T, \underline{w}^T, w_n^T]^T$. Consequently,

$$\begin{aligned} \lim_{t\to\infty} y(t) &= \lim_{t\to\infty} (e_1(t) + \tilde{r}(t) + y_*) \\ &= \lim_{t\to\infty} e_1(t) + \lim_{t\to\infty} \tilde{r}(t) + y_* \\ &= y_*, \end{aligned} \tag{55}$$

which confirms the convergence result (8).

The proof that there exists no Zeno behavior, i.e.

$$\inf_{k,\underline{k}\in\mathbb{Z}_+} \{t_{k+1} - t_k, \underline{t}_{\underline{k}+1} - \underline{t}_{\underline{k}}\} > 0, \tag{56}$$

follows arguments similar to those in [39, Theorem 3] and is therefore omitted. □

4.3. *Two relaxations of Assumption 3.2*

We now consider a more general case in which Assumption 3.2 is relaxed as follows.

Assumption 4.8. For each $j = 1, \ldots, n$, there exist a set $\Omega \subseteq \mathbb{R}^m$, nonnegative constants M_{j1} and M_{j2}, and functions $\psi_{jb} \in$

$\mathcal{K}_\infty$ for $b = 1, \dots, j$, which are linearly bounded near zero and continuously differentiable on $(0, \infty)$, such that $y_* \in \Omega$,

$$|f_j(y, 0, \dots, 0)| \leq M_{j1} \tag{57}$$

and

$$\begin{aligned}|f_j(x_1, x_2, \dots, x_j, d_j) - f_j(y, 0, \dots, 0)| \\ \leq \psi_{j1}(|x_1 - y|) + \sum_{b=2}^{j} \psi_{jb}(|x_b|) + \psi_j^d(|d_j|) + M_{j2},\end{aligned} \tag{58}$$

for all $[x_1^T, \dots, x_j^T]^T \in \mathbb{R}^{jm}$, $d_j \in \mathbb{R}^{p_j}$, and $y \in \Omega$.

Under Assumption 4.8, the functions f_j's are not required to vanish at the equilibrium point, making this formulation more general than Assumption 3.2. In this case, we obtain

$$\begin{aligned}&|f_j(x_1, x_2, \dots, x_j, d_j)| \\ &= |f_j(x_1, x_2, \dots, x_j, d_j) - f_j(y, 0, \dots, 0) + f_j(y, 0, \dots, 0)| \\ &\leq \psi_{j1}(|e_1|) + \sum_{b=2}^{j} \psi_{jb}(|x_b|) + \psi_j^d(|d_j|) + M_j,\end{aligned} \tag{59}$$

where $M_j = M_{j1} + M_{j2}$. Following the same reasoning as in Sec. 4.1, Lemmas 4.2–4.6 remain valid, with (17) replaced by

$$\begin{aligned}|e_j(t)| \leq \max_{b=1,2,\dots,j+1}\{e^{-l_j t}|e_j(0)|, \gamma_{e_j}^{e_b}(\|e_b\|_\infty), \gamma_{e_j}^{\dot{r}}(\|\dot{r}\|_\infty), \\ \gamma_{e_j}^{w_n}(\|w_n\|_\infty), \gamma_{e_j}^{\bar{d}_j}(\|\bar{d}_j\|_\infty), \gamma_{e_j}^{\bar{M}_j}(\bar{M}_j)\},\end{aligned} \tag{60}$$

where $\bar{M}_j = \sum_{b=1,\dots,j} M_b$ and $\gamma_{e_j}^{\bar{M}_j} \in \mathcal{K}_\infty$ is a Lipschitz function linearly bounded near zero.

Furthermore, by the cyclic small-gain theorem (Theorem 2.1 in this paper), if the conditions of Theorem 4.7 hold with Assumption 3.2 replaced by Assumption 4.8, then (54) becomes

$$|X(t)| \leq \max\{\beta(|X(0)|, t), \gamma_x^d(\|d\|_\infty), \gamma_x^{\bar{M}}(\bar{M}_n)\}, \tag{61}$$

for some $\beta \in \mathcal{KL}$ and $\gamma_x^d, \gamma_x^{\bar{M}} \in \mathcal{K}_\infty$. This implies that $y(t)$ converges to a neighborhood of y_*, whose radius depends on $\|d\|_\infty$ and $\bar{M}_n$. In particular, if $\lim_{t\to\infty} d(t) = 0$ and $\bar{M}_n = 0$, this neighborhood collapses to the single point y_*.

Assumption 3.2 can also be relaxed to the following case with unknown parameters.

Assumption 4.9. The function f_1 satisfies all conditions in Assumption 3.2. For each $j = 2, \dots, n$, there exist functions $\psi_{jb} \in \mathcal{K}_\infty$ for $b = 2, \dots, j$, which are linearly bounded near zero and continuously differentiable on $(0, \infty)$, and unknown constant $\varpi_j > 0$, such that

$$|f_j(x_1, \dots, x_j, d_j)| \leq \sum_{b=2}^{j} \varpi_j \psi_{jb}(|x_b|),$$

for all $[x_1^T, \dots, x_j^T]^T \in \mathbb{R}^{jm}$ and $d_j \in \mathbb{R}^{p_j}$.

Under Assumption 4.9, the growth bounds of the uncertain dynamics are scaled by an unknown factor ϖ_j. In this setting, the small-gain framework developed in this paper can be combined with adaptive designs as in [38, 47–52]. By introducing suitable adaptive laws, the unknown constant ϖ_j can be compensated online, and prior knowledge of the constants ω (strong monotonicity) and ϑ (Lipschitz-continuous gradient) of the objective function is no longer required. This integration preserves the small-gain analysis while enabling fully adaptive implementation.

5. Application to a Source-Seeking Problem

We apply the proposed event-triggered feedback optimization method to a two-dimensional source-seeking task. Consider a robot moving in the XY-plane with dynamics [45]

$$\dot{x}_1 = x_2, \tag{62}$$

$$\dot{x}_2 = f_2(x_2)\theta + u, \tag{63}$$

$$y = x_1, \tag{64}$$

where $x_1 = [x_{11}, x_{12}]^T$, $x_2 = [x_{21}, x_{22}]^T$ and $u = [u_1, u_2]^T \in \mathbb{R}^2$ represent the position, velocity, and control torque, respectively. The unknown friction coefficient vector $\theta = [\theta_1, \theta_2]^T$ satisfies $0.01 \leq \theta_i \leq 0.1$ for $i = 1, 2$, and $f_2(x_2) = \text{diag}(f_{21}(x_{21}), f_{22}(x_{22}))$ with $f_{2i}(x_{2i}) = -|x_{2i}|x_{2i}$ for $i = 1, 2$.

The robot aims to locate an unknown signal source at an unknown point $y^* \in \mathbb{R}^2$, that is

$$\lim_{t\to\infty} y(t) = y^*. \tag{65}$$

The signal strength at $y \in \mathbb{R}^2$ is modeled by

$$F(y) = \frac{\delta_1}{|y - y^*|^2 + \delta_2}, \tag{66}$$

where $\delta_1, \delta_2 > 0$ are unknown constants determined by the environment. Although the robot does not know y^* or $F(\cdot)$, it can measure $F(y)$ and its gradient $\nabla F(y)$ in real time.

Define

$$c(y) = \frac{\bar{\delta}}{F(y)}, \tag{67}$$

where $\bar{\delta} > 0$ is a design constant. Then y^* is the minimizer of

$$\min\ c(y), \tag{68}$$

and thus the source-seeking objective (65) is equivalent to the optimization goal (8). From the real-time data $F(y)$ and $\nabla F(y)$, we obtain

$$\nabla c(y) = -\frac{\bar{\delta}}{F^2(y)} \nabla F(y). \tag{69}$$

For simulation, the parameters are selected as $\delta_1 = 0.0042$, $\delta_2 = 0.001$, $\theta = [0.04, 0.06]^T$ and $y^* = [1, 2]^T$.

With $\bar{\delta} = 0.004$, the function $c(y)$ satisfies the strong monotonicity condition with $\omega = 0.8$. Assumptions 3.2 and 3.3 hold with $\psi_{22}(\tau) = 0.1\tau^2$ and $\vartheta = 1$ for $\tau \in \mathbb{R}_+$.

Following the design procedure in Sec. 4, the feedback optimization protocol (11) and event-triggering mechanism (12) are implemented with

$$\rho = \frac{0.2}{\vartheta}, \tag{70}$$

$$\alpha_1(\xi) = -a_1\xi, \tag{71}$$

$$\alpha_2(\xi) = -(a_{22}|\xi| + a_{21})\xi, \tag{72}$$

$$\gamma_{w_2}^{e_2}(\tau) = \frac{3\vartheta}{4}\tau, \tag{73}$$

$$\gamma_{\underline{w}}^{c}(\tau) = \frac{\omega}{16\vartheta}\tau, \tag{74}$$

for $\xi \in \mathbb{R}^2$ and $\tau \in \mathbb{R}$, where $a_1 = 1.1 + l_1 + \frac{10}{9}\max\{3\rho\vartheta, 9\rho\frac{\vartheta^2}{\omega}\}$, $a_{22} = \frac{20}{81}a_1^2 + 0.2$, $a_{21} = \frac{10}{9}a_1^2 + a_1 + l_2 + \frac{10}{9}a_1\max\{3\rho\vartheta, 9\rho\frac{\vartheta^2}{\omega}\}$, $l_1 = 1$, and $l_2 = 0.1$.

With initial conditions $x_1(0) = [0,4]^T$, $x_2(0) = [0,0]^T$, and $r(0) = [0,4]^T$, the trajectories of the robot's position y and control input u are shown in Figs. 3 and 4. The results show that $y(t)$ converges asymptotically to $[1,2]^T$, while $u(t)$ approaches zero. Figure 5 indicates that the sampled gradient $\nabla c(y^s)$ converges to zero, consistent with the fact that $[1,2]^T$ corresponds to the source location. As shown in Figs. 6 and 7, the minimum inter-event times of $\{t_k : k \in \mathbb{Z}_+\}$ and $\{t_{\underline{k}} : \underline{k} \in \mathbb{Z}_+\}$ are no less than 0.002 and 0.1, respectively. These observations align with Theorem 4.7, confirming the convergence and Zeno-free behavior of the proposed design.

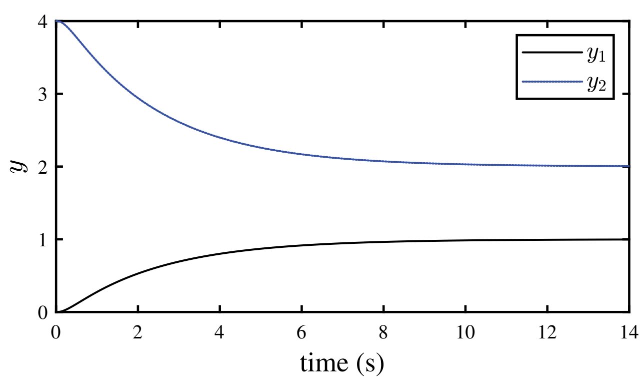

Fig. 3. Trajectory of the position y of the robot.

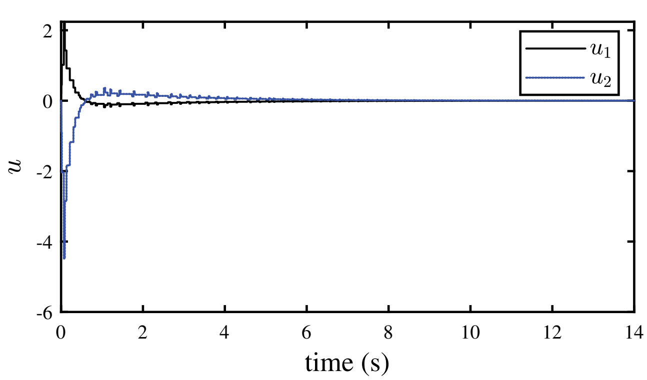

Fig. 4. Trajectory of the control torque u of the robot.

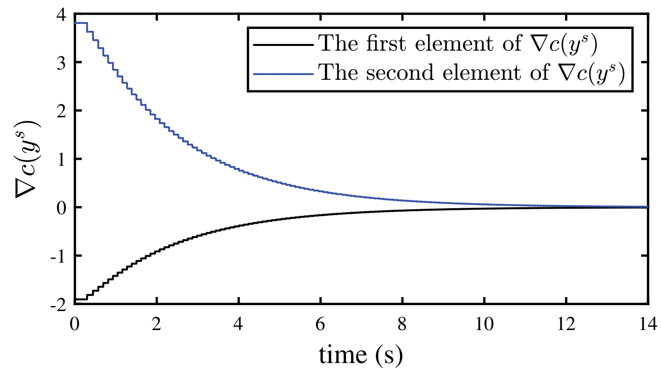

Fig. 5. Trajectory of the sampled gradient $\nabla c(y^s)$.

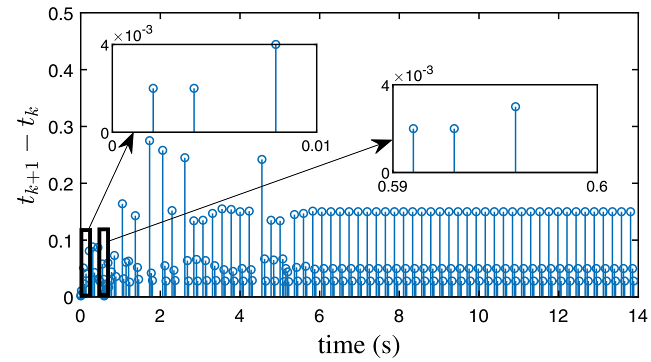

Fig. 6. Evolution of the inter-event times of $\{t_k : k \in \mathbb{Z}_+\}$.

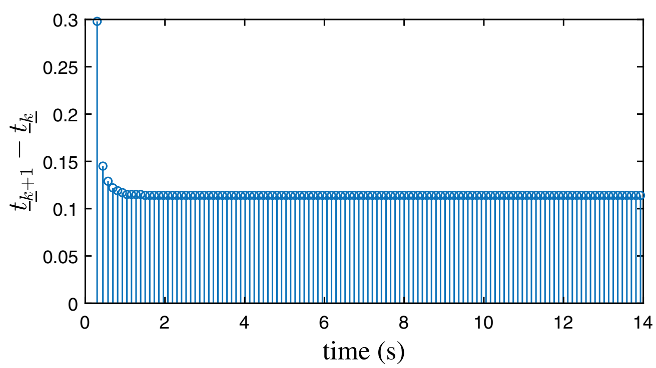

Fig. 7. Evolution of the inter-event times of $\{t_{\underline{k}} : \underline{k} \in \mathbb{Z}_+\}$.

6. Conclusions

This paper has investigated the event-triggered feedback optimization problem for a class of uncertain nonlinear systems in strict-feedback form. A novel event-triggered feedback optimization protocol was developed that relies on the sampled system states and the real-time gradients of the objective function along the system output trajectories. It was shown that the closed-loop dynamics can be represented as a feedback interconnection of multiple ISS subsystems. By applying the small-gain theorem, sufficient small-gain conditions were established to ensure closed-loop stability and guarantee convergence of the system output to the optimal solution. Moreover, the proposed design excludes Zeno behavior. Finally, we discussed two extensions for systems whose uncertain dynamics are nonvanishing at the equilibrium point and are bounded by nonlinear functions involving unknown parameters, thereby demonstrating the broader applicability of the proposed framework. The proposed approach was further applied to a source-seeking problem, thereby verifying its effectiveness through simulation results.

The small-gain method established in this work provides an inherent potential to handle time delays, disturbances, and measurement errors systematically. Investigating these scenarios is therefore a promising direction for future work.

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